Chapter 3

Astounding Analysis

3.1 Discontinuities

Vertical Asymptotes occur when factors in the denominator $= 0$ and do not cancel with factors in the numerator.

- **Vertical asymptotes** are vertical lines the graph approaches.
- The equation of the vertical asymptote is $x = (\text{that number which makes the denominator } = 0)$

Holes (**Removable Discontinuities**) occur when the factor in the denominator $= 0$ and it cancels with like factors in the numerator.

- Holes are open "points" so they have an x and y coordinate.
- The x-value is the number that makes the cancelled factor $= 0$.
- The y-value is found by substituting x into the "reduced" equation (after cancelling) like factors.

Find the vertical asymptotes and holes (if any) for the following. Don’t forget that vertical asymptotes are equations and holes are points!

**Example 3.1**

$y = \frac{1}{x}$

Vertical Asymptote: $x = 0$
Hole: None

**Example 3.2**

$y = \frac{x(x-1)}{x-1}$

Vertical Asymptote: None
Hole: (1,1) since (x-1) was cancelled, the hole is at x=1. To find the y-coordinate, plug 1 into the reduced equation: \( \frac{x(x-1)}{x-1} = x = 1 \)

**Exercise 3.1**

$y = \frac{4x+3}{x-7}$  \hspace{1cm} \text{(Solution on p. 31.)}

**Exercise 3.2**

$y = \frac{x}{x-2x}$

**Exercise 3.3**

$y = \frac{7}{(x-9)(x+1)}$  \hspace{1cm} \text{(Solution on p. 31.)}

**Exercise 3.4**

$y = \frac{7x}{2x^{2}-x+3}$  \hspace{1cm} \text{(Solution on p. 31.)}

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\(^1\)This content is available online at <http://cnx.org/content/m13605/1.3/>. 
Exercise 3.5
\[ y = \frac{2x+1}{(x+5)^2} \]  
(Solution on p. 31.)

Exercise 3.6
\[ y = \frac{x^2+3}{x^2+25} \]  
(Solution on p. 31.)

Exercise 3.7
\[ y = \frac{x^2-7}{x^2+2} \]  
(Solution on p. 31.)

Exercise 3.8
\[ y = \frac{5}{|x-3|} \]  
(Solution on p. 31.)

Exercise 3.9
\[ y = \frac{4}{|x-1|} \]  
(Solution on p. 31.)

Exercise 3.10
\[ y = \frac{3(x^2-x-6)}{4(x^2-9)} \]  
(Solution on p. 31.)

Exercise 3.11
\[ y = \frac{-2(x^2-4)}{3(x^2+4x+4)} \]  
(Solution on p. 31.)

Exercise 3.12
\[ y = \frac{x-4}{x+2} \]  
(Solution on p. 31.)

Exercise 3.13
\[ y = \frac{x^2(x-3)}{x^2-3x} \]  
(Solution on p. 31.)

Exercise 3.14
\[ y = \frac{x^2-1}{x-1} \]  
(Solution on p. 32.)

Exercise 3.15
\[ y = \frac{2x^2-3x-5}{x^2-1} \]  
(Solution on p. 32.)

3.2 Horizontal Asymptotes

Horizontal asymptotes are horizontal lines the graph approaches.

Horizontal Asymptotes CAN be crossed.

To find horizontal asymptotes:

- If the degree (the largest exponent) of the denominator is bigger than the degree of the numerator, the horizontal asymptote is the x-axis (y = 0).
- If the degree of the numerator is bigger than the denominator, there is no horizontal asymptote.
- If the degrees of the numerator and denominator are the same, the horizontal asymptote equals the leading coefficient (the coefficient of the largest exponent) of the numerator divided by the leading coefficient of the denominator.

One way to remember this is the following mnemonic device: BOBO BOTN EATS DC

- BOBO - Bigger on bottom, y=0
- BOTN - Bigger on top, none
- EATS DC - Exponents are the same, divide coefficients

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\(^2\)This content is available online at <http://cnx.org/content/m13606/1.8/>. 
Find the Horizontal Asymptotes of the following:

Exercise 3.16  \( f(x) = \frac{4x}{x^3} \) (Solution on p. 32.)

Exercise 3.17  \( g(x) = \frac{5x^2}{x^2+2} \) (Solution on p. 32.)

Exercise 3.18  \( h(x) = \frac{4x^2}{(x-2)(x+4)} \) (Solution on p. 32.)

Exercise 3.19  \( g(x) = \frac{6}{(x+3)(4-x)} \) (Solution on p. 32.)

Exercise 3.20  \( f(x) = \frac{3x(x-1)}{2x^2-5x+3} \) (Solution on p. 32.)

Exercise 3.21  \( q(x) = \frac{(-x)(1-x)}{3x^2+5x-2} \) (Solution on p. 32.)

Exercise 3.22  \( r(x) = \frac{2}{(x-8)^3} \) (Solution on p. 32.)

Exercise 3.23  \( r(x) = \frac{x}{x-1} \) (Solution on p. 32.)

Exercise 3.24  \( g(x) = \frac{x-3}{x^2+1} \) (Solution on p. 32.)

Exercise 3.25  \( r(x) = \frac{4x^2+x}{x^2+4} \) (Solution on p. 32.)

3.3 Slant Asymptotes

Just like vertical and horizontal asymptotes, slant asymptotes are lines the graph approaches. They are also called oblique asymptotes.

A graph has a slant asymptote if the degree of the numerator is bigger than the degree of the denominator (there is no horizontal asymptote).

To find slant asymptotes, divide the numerator by the denominator and keep only the quotient (the answer, throw away the remainder). Don’t forget that these are still lines, so they are written as \( y = \).

To divide, you either have to use long division or synthetic division (if possible).

PRACTICE - Find the Slant Asymptotes:

Exercise 3.26  \( y = \frac{3x^3}{x^2-1} \) (Solution on p. 32.)

Exercise 3.27  \( y = \frac{2x^2}{x+1} \) (Solution on p. 32.)

Exercise 3.28  \( y = \frac{x^2-9x+2}{x+4} \) (Solution on p. 32.)

Exercise 3.29  \( y = \frac{x^2-37}{x+3} \) (Solution on p. 32.)

Exercise 3.30  \( y = \frac{2x^3+7x^2-4}{(x+3)(x-1)} \) (Solution on p. 32.)

\(^a\)This content is available online at <http://cnx.org/content/m13608/1.1/>.
Exercise 3.31
\[ y = \frac{x^2+5x+8}{x+3} \]  
(Solution on p. 33.)

Exercise 3.32
\[ y = \frac{2x^2+x}{x+1} \]  
(Solution on p. 33.)

Exercise 3.33
\[ y = \frac{(2x)(x+11)}{x-4} \]  
(Solution on p. 33.)

Exercise 3.34
\[ y = \frac{x^3}{(x-1)^3} \]  
(Solution on p. 33.)

Exercise 3.35
\[ y = \frac{x^4-x+3}{x^2+x-2} \]  
(Solution on p. 33.)