

$$e^{i\pi} + 1 = 0$$

Introduction

Leonhard Euler, who was born in Bâle, Switzerland, on April 15, 1707, and died in St. Petersburg, Russia, on September 7, 1783, is recognized as having been one of the most influential mathematicians of all times. He is credited with forming the branch of mathematics called analysis, initiating the use of the sigma symbol for sums, using the notation $f(x)$ for function values, adding proofs of theorems which were hitherto unknown, and is recognized for making major contributions to the study of optics, mechanics, electricity and magnetism, and differential equations.

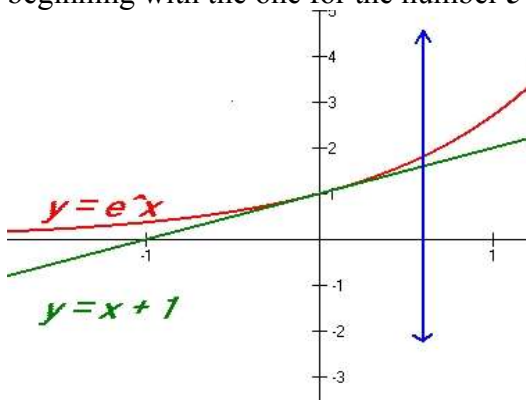


He is first to have shown that any complex number raised to a complex power is equal to a complex number. In number theory he was first to establish the divergence to infinity of the harmonic series. Additionally, Euler proved that the Binomial Theorem is valid for rational powers, not just for integer exponents. The first letter of his last name is the reason that the base of the Natural Logarithm is e .

A major contribution Euler made to pure mathematics was the establishment of the intimate relationship between the number, e , the square root of minus 1, i , and the sine and cosine functions. This article is devoted to Euler's most famous equation in which the number π is related to e and i together with 0 and 1 .

Three Important Infinite Series

There are three exceptionally important infinite series which we need to establish, beginning with the one for the number e defined by the equation $\ln e = 1$. We start with



the graph of $y = e^x$, and its tangent line $y = x + 1$. The blue vertical line passes through the arbitrary point $(x, 0)$. Clearly the area bounded between $y = e^x$, the x -axis, the y -axis, and the blue line is quite closely approximated by the area bounded within $y = x + 1$, the x -axis, the y -axis, and the blue line. This leads to

$$\int_0^x e^t dt \approx \int_0^x (t + 1) dt \quad (1)$$

Integrating, (1) leads to

$$\begin{aligned} e^t \Big|_0^x &\approx \left(\frac{1}{2}t^2 + t \right) \Big|_0^x \\ e^x - 1 &\approx \frac{1}{2}x^2 + x \\ e^x &\approx 1 + x + \frac{1}{2}x^2 \end{aligned} \tag{2}$$

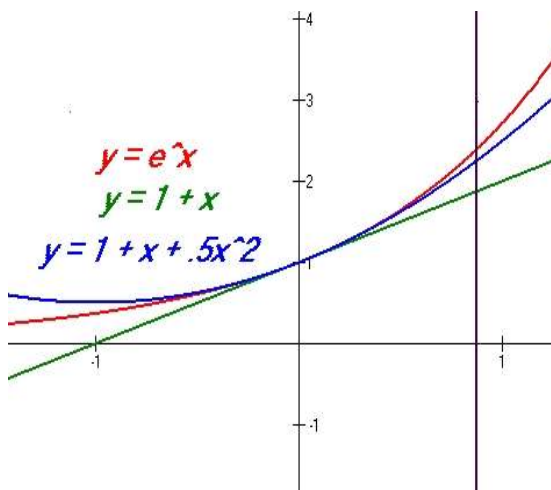
The approximation (2) appears to be built from the tangent line approximation

$$e^x \approx 1 + x$$

with the addition of the $\frac{1}{2}x^2$ term, incidentally. We continue to build on (2). Notice

how the graph of $y = 1 + x + \frac{1}{2}x^2$, as

shown in blue, is closer to $y = e^x$ for values of x near 0. The vertical line in black intersects the x -axis at the point $(x, 0)$ as before. Again we can use the fact that the area under the exponential curve is approximated by the area under the blue parabola in setting up two integrals



$$\int_0^x e^t dt \approx \int_0^x \left(1 + t + \frac{1}{2}t^2 \right) dt$$

leading to a better polynomial

approximation for the exponential function. Carrying out the integration we obtain

$$e^t \Big|_0^x \approx \left(t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 \right) \Big|_0^x$$

Adding 1 to both sides we have the new approximation

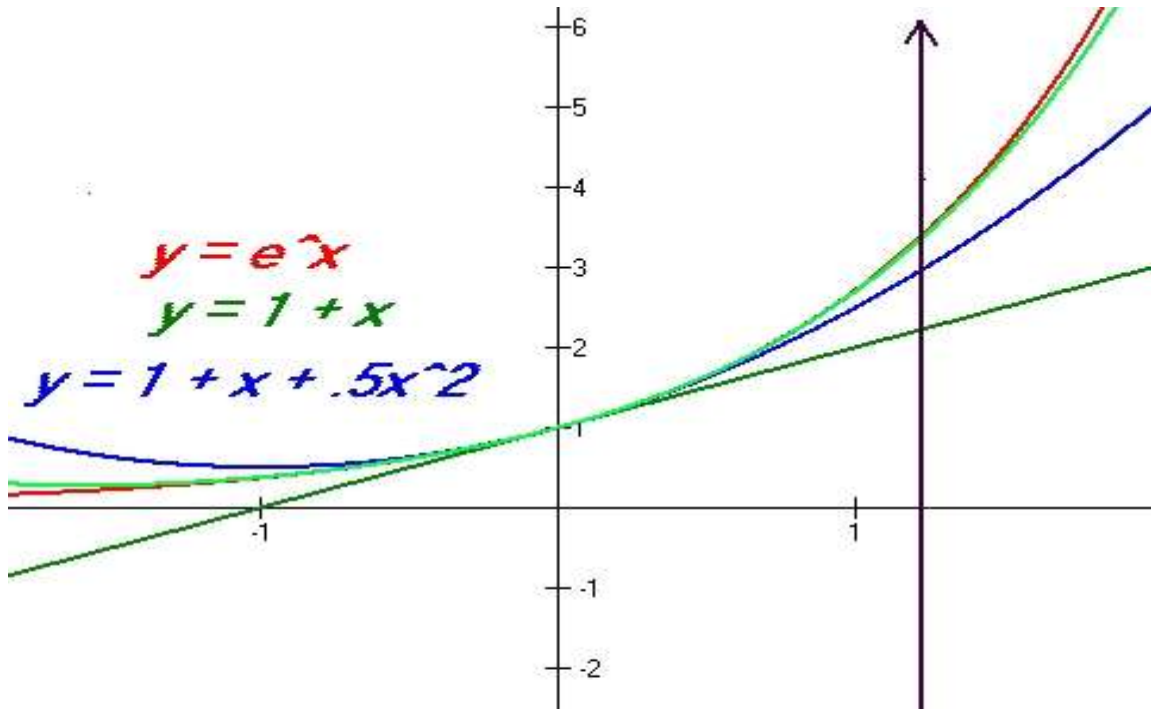
$$e^x \approx 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \tag{3}$$

which is the same as (2) but with the added cubic term.

Changing the x 's in (3) to t 's and integrating again from 0 to x yields

$$e^x \approx 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$

The next page shows the graph of the 4th degree polynomial approximation added in light green. Notice how closely it hugs the red exponential function's graph for values of x



which are in the interval between -1 and 1. We can continue integrating as before producing higher degree polynomial approximations. Since it is true that

$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \text{ for } x \in (-\infty, \infty) \quad (4)$$

and the approach to 0 is rapid, we have arrived at the following infinite series expansion for e^x .

$$e^x \approx 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n + \dots$$

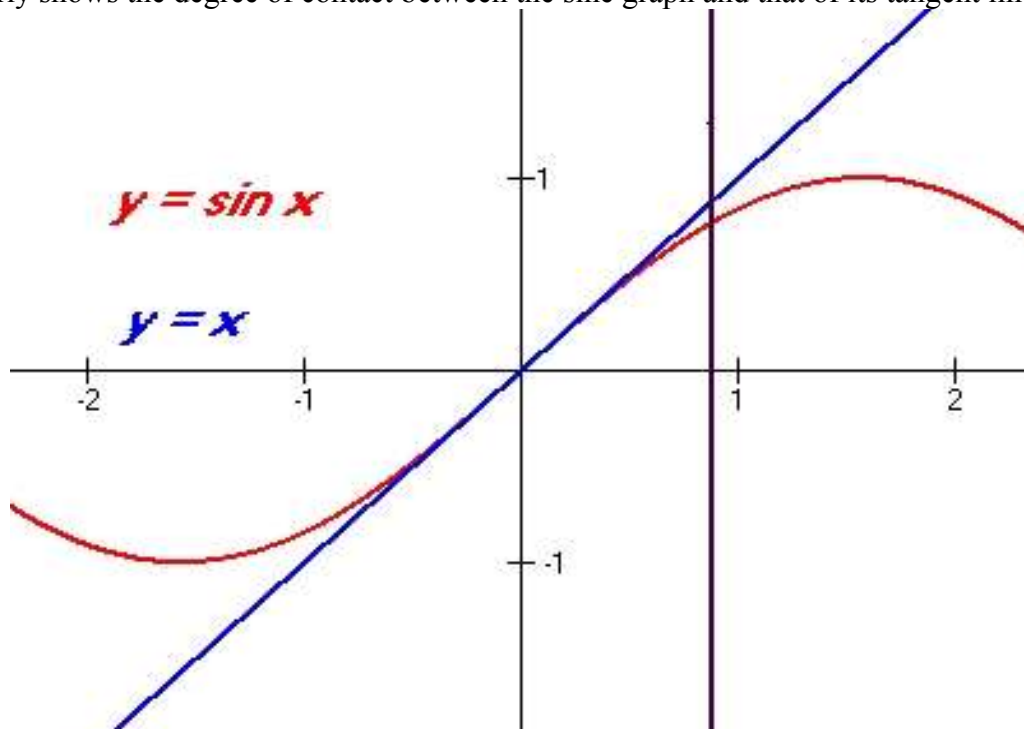
This can be written more compactly as

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (5)$$

Series (5) converges to e^x since no matter how large an x is chosen, the limit in (4) implies that the series eventually settles to a limit. This is not a precise proof. For that you will have to wait until you study Calculus BC.

We are now ready to turn to the second and third important series. The method used to get the second automatically produces the third. Thus we literally get two series for the price of getting one. We start with the sine function and examine it closely as it passes through the origin. The derivative of $\sin x$ is $\cos x$ so that if $x = 0$, $\cos 0 = 1$ tells us that the sine function is tangent to the line $y = x$ as it passes through $(0, 0)$. The next graph

clearly shows the degree of contact between the sine graph and that of its tangent line.



The graph is large to show the relatively slow divergence between the sine and the line $y = x$ for values of x between -1 and 1 . Thus we can write the first approximation as

$$\sin x \approx x \quad (6)$$

While this may seem a little crude, consider, for example,

$$\sin(0.00006487) = .00006486999995$$

Impressive as this seems, we actually can obtain an approximation which will produce a value of $\sin x$ for $x = 1000$ which is *just as close to being correct!*

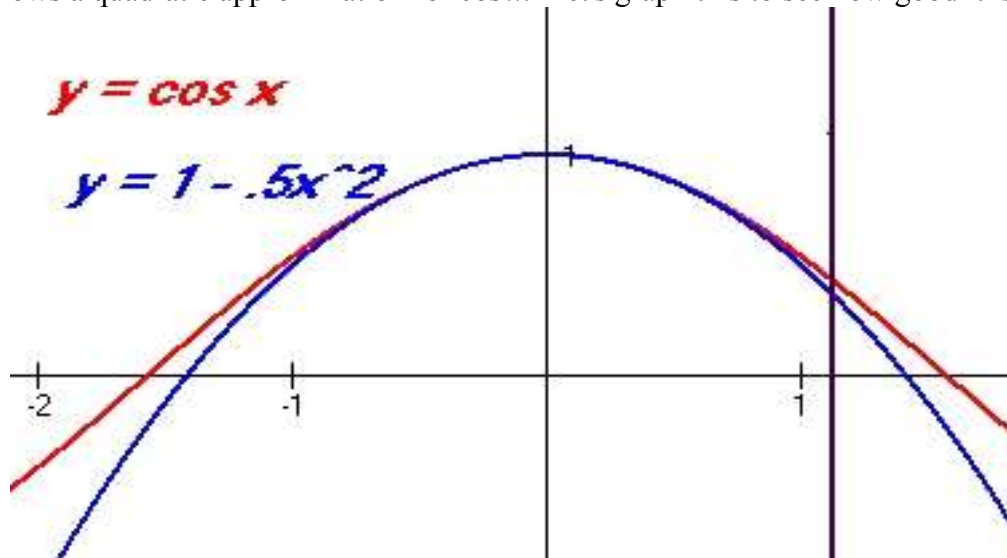
We change the variable in (6) to t and integrate both sides from 0 to x , where x is an arbitrary value. The idea being that the areas under the graph of the sine function and under $y = x$ are nearly the same between the y -axis and a vertical line through the point $(x, 0)$.

$$\int_0^x \sin t \, dt \approx \int_0^x t \, dt \Rightarrow -\cos t \Big|_0^x = \cos t \Big|_x^0 \approx \frac{1}{2}t^2 \Big|_0^x$$

Evaluating and rearranging terms we find

$$\cos x \approx 1 - \frac{1}{2}x^2 \quad (7)$$

(7) shows a quadratic approximation for $\cos x$. Let's graph this to see how good it is.



There appears to be a rather tight fit between the cosine and the parabola. For example,

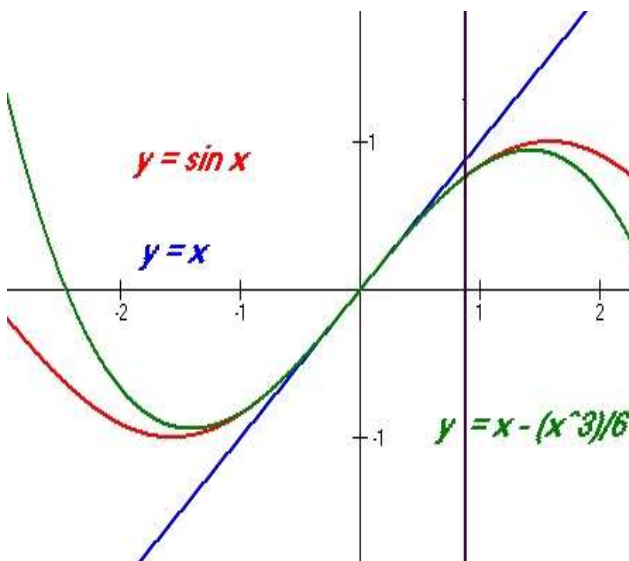
$$\cos(.4) = .921060994 \quad \text{while} \quad 1 - \frac{1}{2}(.4)^2 = .920000000$$

so for an x as comparatively large as 0.4, the parabolic approximation to the cosine is not too bad at all!

Now watch what happens when we take (7), change x to t , and integrate from 0 to x .

$$\int_0^x \cos t \, dt \approx \int_0^x \left(1 - \frac{1}{2}t^2\right) dt \Rightarrow \sin t \Big|_0^x \approx \left(t - \frac{1}{3!}t^3\right) \Big|_0^x$$

$$\sin x \approx x - \frac{1}{3!}x^3 \tag{8}$$



Notice how (8) builds upon (6) in approximating the sine function, this time with a cubic polynomial. The graph at the left shows how approximation (8), shown in green, improves upon approximation (6), seen in red. For the portion of the sine graph shown in the graph, the concavity of the cubic polynomial matches correctly, being upward for positive values of x and downward for negative values of x . Let's consider a numerical example to

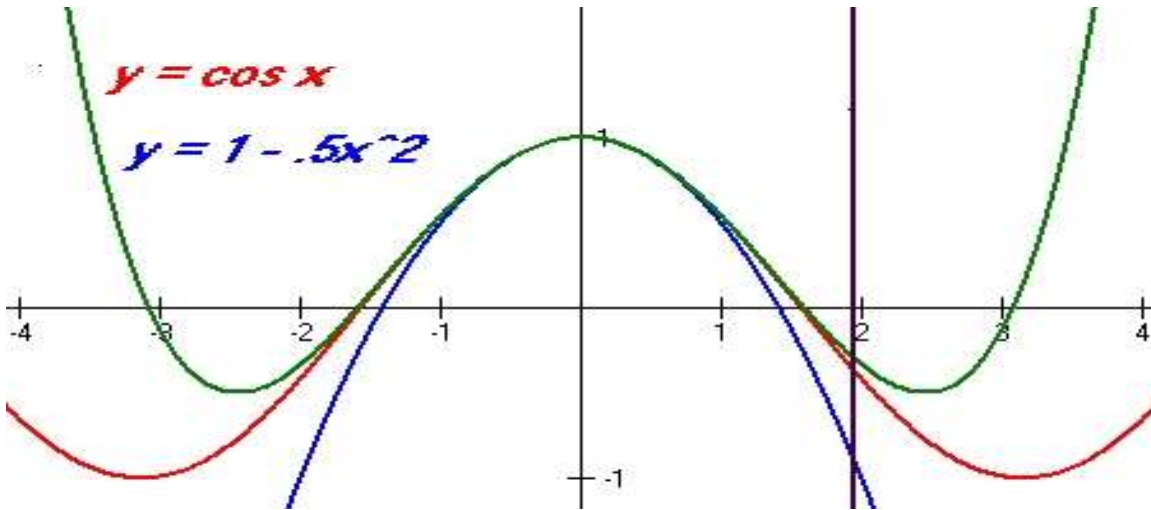
demonstrate the accuracy of (8).

$$\sin(.4) = .3894183423 \quad \text{while} \quad .4 - \frac{1}{6}(.4)^3 = .3893333333$$

Taking (8), changing x to t and integrating from 0 to x produces the approximation

$$\cos x \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 \quad (9)$$

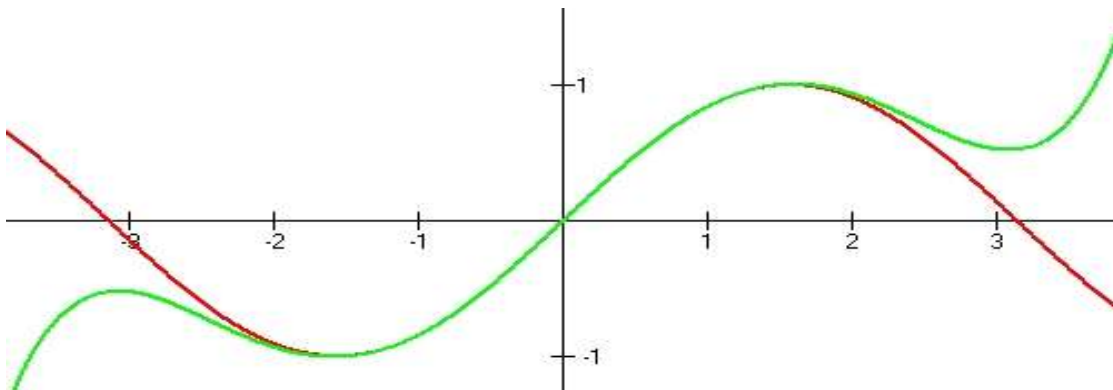
The closeness of the 4th degree polynomial to the cosine function can be observed in the graph below. The green polynomial closely hugs the cosine graph.



Processing (9) in the same way we processed (8), the result is

$$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \quad (10)$$

Below is a graph showing the fit of the 5th degree polynomial to the sine function. Only

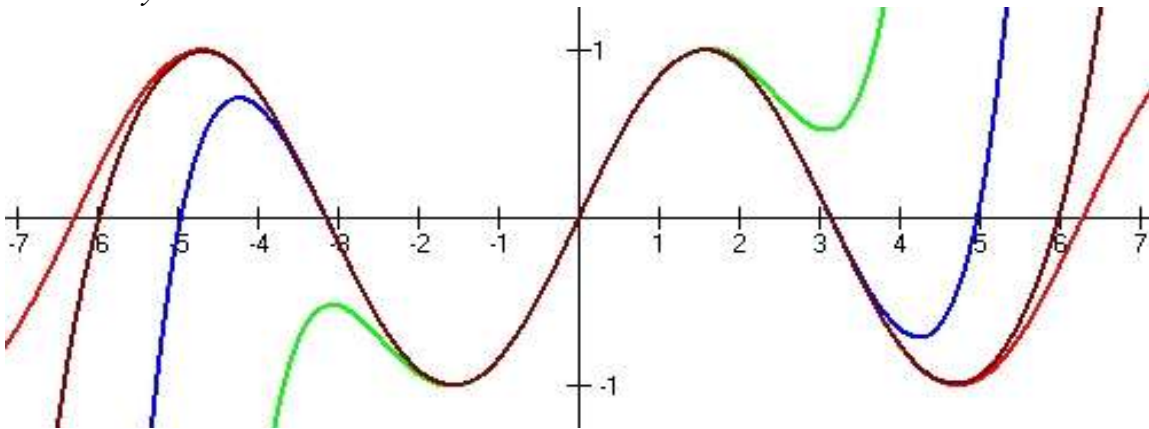


the most hard-boiled person cannot see how impressive these approximating polynomials are becoming as each added term effectuates a spectacular improvement.

Looking at the approximations (6), (8), and (10) you clearly see the pattern. It appears that

$$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \dots \quad (11)$$

provides a means of approximating $\sin x$ as closely as we please by increasing the degree of the polynomial. For the green 9th degree polynomial (the 5 terms in the right side of (11)) the graph below shows the fit continues to improve further away from the y-axis with the blue 11th degree polynomial being even better. The maroon polynomial is the 13th degree polynomial which appears to provide a good fit for nearly one period on either side of the y-axis!



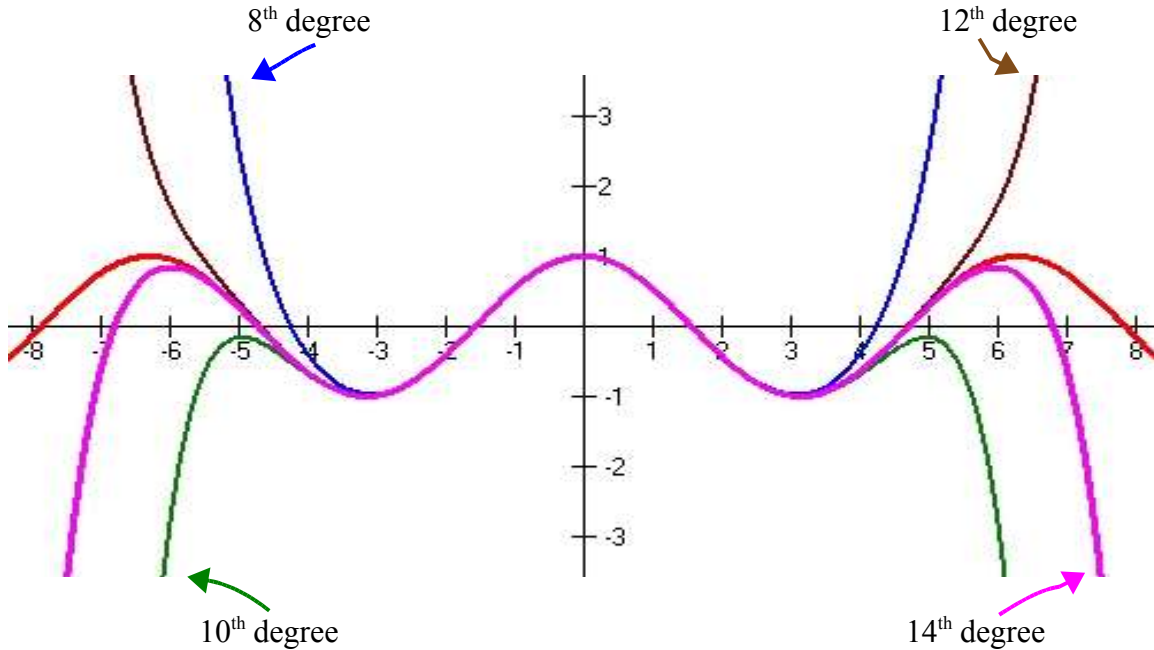
The approximation in (11) together with the limit in (4) (with the *rapid* approach to 0 of the n th term) can be generalized to provide an infinite series expansion for $\sin x$ which converges exactly for any real number value of x .

$$\sin x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} x^{(2n-1)} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \dots \quad (12)$$

The pattern of the approximations for the cosine function given in (7) and (9) gives rise to the infinite series for the cosine as follows.

$$\cos x = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-2)!} x^{(2n-2)} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \dots \quad (13)$$

To show the evolution of accuracy for a few of the approximating polynomials contained within (13) of specific degrees, study the graph on the next page. The red graph is the cosine curve itself. It is completely (due to the close approximation) covered by all the approximating polynomials between -3 and $+3$ radians. The 14th degree approximating polynomial, whose graph is pink, remains in the very near vicinity of the cosine curve throughout the interval from -6 to $+6$ radians.



Fittingly, the series of terms for the sine function contains only odd-powered x 's while the series for the cosine function has only even powers of x . If you differentiate with respect to x the series for $\sin x$ the result, term-by-term, will be the series for the $\cos x$, while it is easy to see that the derivative with respect to x of the series for $\cos x$ will produce, term-by-term, the *opposite* of the series for $\sin x$. This is, of course, as it should be. It is just so satisfyingly good to point this out.

The Magic Euler Equation

No doubt you have already noticed that all of the terms in both series (12) and (13) are contained within the series for e^x , which is series (5). This is remarkable, and caused quite a stir when this fact first came to the light of reason. Mathematics is just packed with such seeming coincidences, a fact which merely serves to enhance the interest mathematics has for us math fans. Euler proved that the series for e^x is not only valid for all real numbers, but that it works for all **complex numbers which have a non-zero i-component!!**

This fact, the convergence of series (5) for all complex x , has profound consequences, one of which leads to the Euler Equation. For the fun of it, let's take series (5) and replace every x with $i\theta$. The result is

$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \frac{1}{6!}(i\theta)^6 + \frac{1}{7!}(i\theta)^7 +$$

$$\frac{1}{8!}(i\theta)^8 + \frac{1}{9!}(i\theta)^9 + \frac{1}{10!}(i\theta)^{10} + \frac{1}{11!}(i\theta)^{11} + \frac{1}{12!}(i\theta)^{12} + \frac{1}{13!}(i\theta)^{13} + \frac{1}{14!}(i\theta)^{14} + \dots$$

You know how to raise $i = \sqrt{-1}$ to integer powers. The result is one of the following: i , -1 , $-i$, or $+1$, repeating in that order, for the exponents $1, 2, 3, 4$, etc., respectively. We are now going to rewrite the series on the preceding page with this in mind, and then re-order them into two groups, one containing i -terms and the other not containing i -terms.

$$e^{i\theta} = 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + i\frac{1}{5!}\theta^5 - \frac{1}{6!}\theta^6 - i\frac{1}{7!}\theta^7 + \frac{1}{8!}\theta^8 + i\frac{1}{9!}\theta^9 - \frac{1}{10!}\theta^{10} - i\frac{1}{11!}\theta^{11} + \frac{1}{12!}\theta^{12} + i\frac{1}{13!}\theta^{13} - \frac{1}{14!}\theta^{14} + \dots$$

$$e^{i\theta} = \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \frac{1}{8!}\theta^8 - \frac{1}{10!}\theta^{10} + \frac{1}{12!}\theta^{12} - \frac{1}{14!}\theta^{14} + \dots\right) + \left(i\theta - i\frac{1}{3!}\theta^3 + i\frac{1}{5!}\theta^5 - i\frac{1}{7!}\theta^7 + i\frac{1}{9!}\theta^9 - i\frac{1}{11!}\theta^{11} + i\frac{1}{13!}\theta^{13} - \dots\right)$$

Now carefully examine this last rewrite.

$$e^{i\theta} = \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \frac{1}{8!}\theta^8 - \frac{1}{10!}\theta^{10} + \frac{1}{12!}\theta^{12} - \frac{1}{14!}\theta^{14} + \dots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + \frac{1}{9!}\theta^9 - \frac{1}{11!}\theta^{11} + \frac{1}{13!}\theta^{13} - \frac{1}{15!}\theta^{15} + \dots\right)$$

The real terms of the above series constitute the series for $\cos \theta$, series (13), while the series having the factor i is the series for $\sin \theta$, series (12). This leads us to an exceedingly important identity:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

The magic equation follows immediately. Just replace θ with π with the following consequence:

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0$$

Adding 1 to both sides yields our goal.

$$e^{i\pi} + 1 = 0$$