

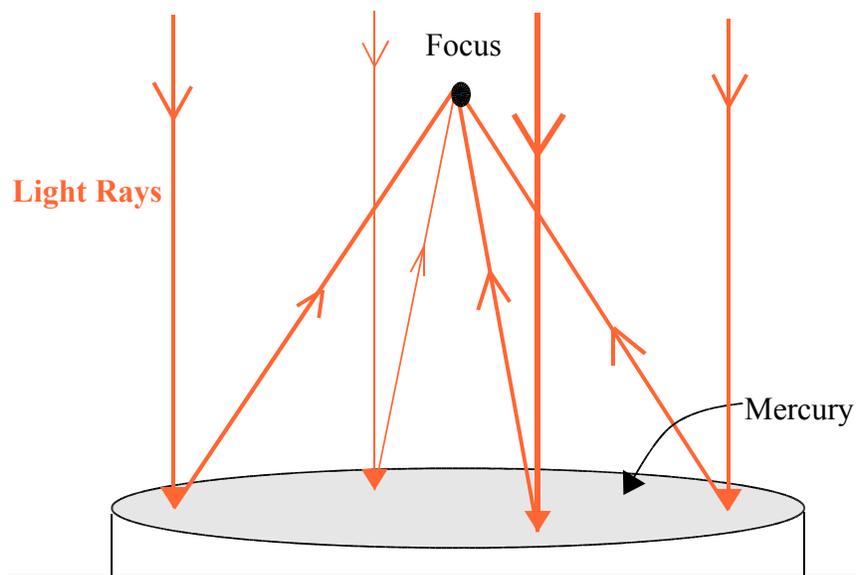
# The Rotating Mercury Reflector

## Introduction

You know that the parabola is the set of all points equidistant from a given fixed point (the focus) and a given fixed line (directrix). If you rotate a parabola about its axis of symmetry a surface called a *paraboloid* is formed. This surface is the set of all points in space equidistant from a fixed point and a fixed plane (the plane of all positions of the directrix as it rotates about its intersection with the axis of the original parabola). Such paraboloids are employed as satellite dishes, flashlight or auto headlight reflectors, and the like.

Here is one of those seemingly “made in heaven” \*coincidences that makes mathematics so fascinating to us math fans. If you rotate a circular container of liquid such as a basin full of water, or even a cup of coffee, the surface of the liquid assumes the shape of a paraboloid!!! The paraboloid is only degraded very near the edge of the container as surface tension disturbs an elegant balance between the centrifugal force of rotation and the force of gravity. The subject of this article is the proof of this fantastic property of rotating liquids.

The title refers to one of the most exquisite applications of this property. Mercury, that liquid metal, is more perfectly reflective than an ordinary mirror (provided oxidation with the air is avoided). A reflective telescope requires a large parabolic mirror to gather light and focus it to a point where a flat mirror sends the light through a few lenses forming the eyepiece of the telescope. Large glass mirrors take months to make and must be very nearly perfect. A rotating mercury reflector telescope contains a large shallow basin rotated at a precise angular velocity which controls the location of the focus.



\* Such coincidences should not surprise us – we do live in a Mathematical Universe.

You can experiment yourself by taking a large basin half-filled with water in a darkened room with a smooth ceiling. Stir the water until you get a good depression at the center. Then take a point source of light, such as a lighted match (be careful) and hold it over the center of the basin. You should then move the light up and/or downward until the light is at the focus. You should see a bright reflection on the ceiling. If you hold the light stationary (watch your fingers), as the rotation of the water gradually slows, the focus moves upward and the image widens unless you raise the light.

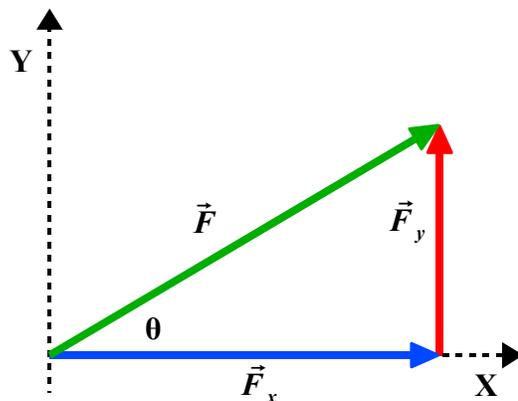
## Some Vector Preliminaries

You need just a tiny amount of background in vectors in order to fully understand why rotating a container of liquid causes the surface to become paraboloidal.

### Definition:

A **vector** is a quantity which embodies both direction and magnitude. The magnitude is a numerical constant called a **scalar**.

The most popular application of the vector concept, the one you need in this topic, is that of a **force**. We use arrows of varying length to represent vectors. The arrow specifies the direction and its length indicates the magnitude. The diagram shows a green



vector with a red vertical component and a blue horizontal component. The components themselves are vectors. The green vector is the sum of the horizontal and vertical vector components. Note that we often place vectors in a coordinate system.

Suppose an object at the origin was “feeling” a force tugging on it as indicated by the green vector. The blue vector indicates the rightward component of that force and the red vector shows the upward component of the force. The angle between

the green and blue vectors,  $\theta$ , is less than  $\frac{\pi}{4}$  since the blue vector is longer than the red one. Therefore more force is exerted to the right than upward by the green vector.

Suppose  $\theta = \frac{\pi}{6}$  ( $30^\circ$ ). If the green vector has a magnitude of 40 pounds, the red

vector would have a magnitude of 20 pounds  $= 40 \sin \frac{\pi}{6}$ . Clearly the blue vector

would have a magnitude of 34.64 pounds  $= 40 \cos \frac{\pi}{6}$ . The vectors form a right

triangle. Thus we expect that  $\sqrt{(34.64)^2 + (20)^2} = 40$ . The diagram shows the

accepted vector notation – bold face capitol letters over which an arrow is drawn.  $\|\vec{F}\|$  is the notation for the magnitude of  $\vec{F}$ . From the viewpoint of vectors, the green vector is the *resultant* of the blue and red vectors. This is indicated by writing

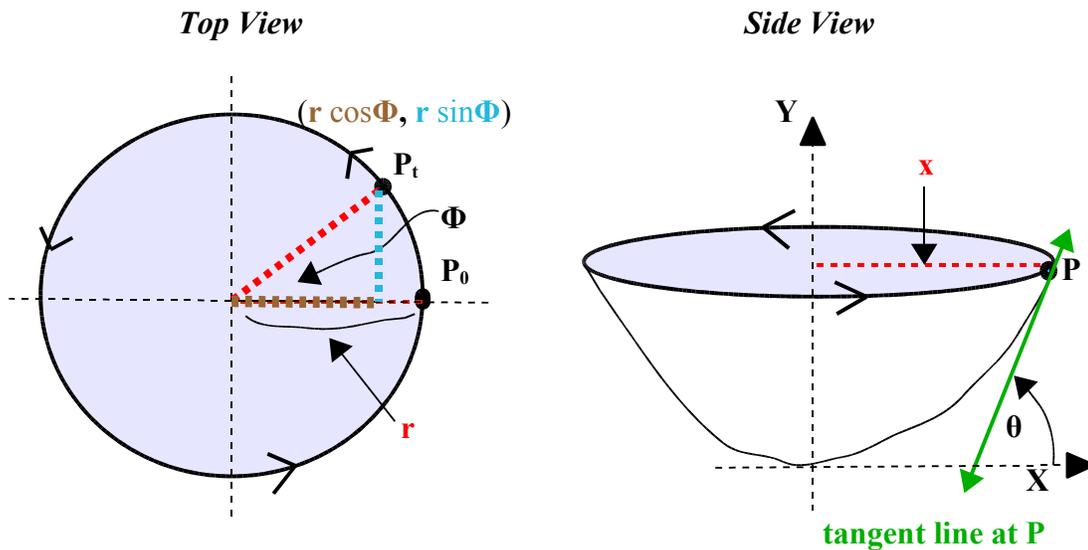
$$\vec{F} = \vec{F}_x + \vec{F}_y$$

This does *not* indicate the relationship between their magnitudes. *That* relationship is

$$\|\vec{F}\| = \sqrt{(\|\vec{F}_x\|)^2 + (\|\vec{F}_y\|)^2}.$$

## The Rotating Liquid

There are two views of the liquid which we need in order to do a proper analysis.



The top view looks directly down on the rotating liquid. Point **P** is on the surface, at the distance **r** from the center of rotation, rotating with the liquid. **P** is subject to two forces: (1) the *centrifugal* force created by the rotation and (2) the force of gravity. We will employ the top view in analyzing the centrifugal force. The side view shows the u-shape of the liquid's surface to which we have drawn a green tangent line at point **P**. In this view we use **x** as the radius, where **x** is the **x**-coordinate of **P** with respect to the axes shown.

## The Centrifugal Force

Let's concern ourselves only with the centrifugal force and the top view for the time being. We are assuming the liquid is rotating at a constant rate which we designate by the symbol  $\omega$  (omega).  $\omega$  is measured in radians per second and specifies the rotational velocity. In the top view diagram we show point **P** rotating from **P**<sub>0</sub> to **P**<sub>t</sub> moving through

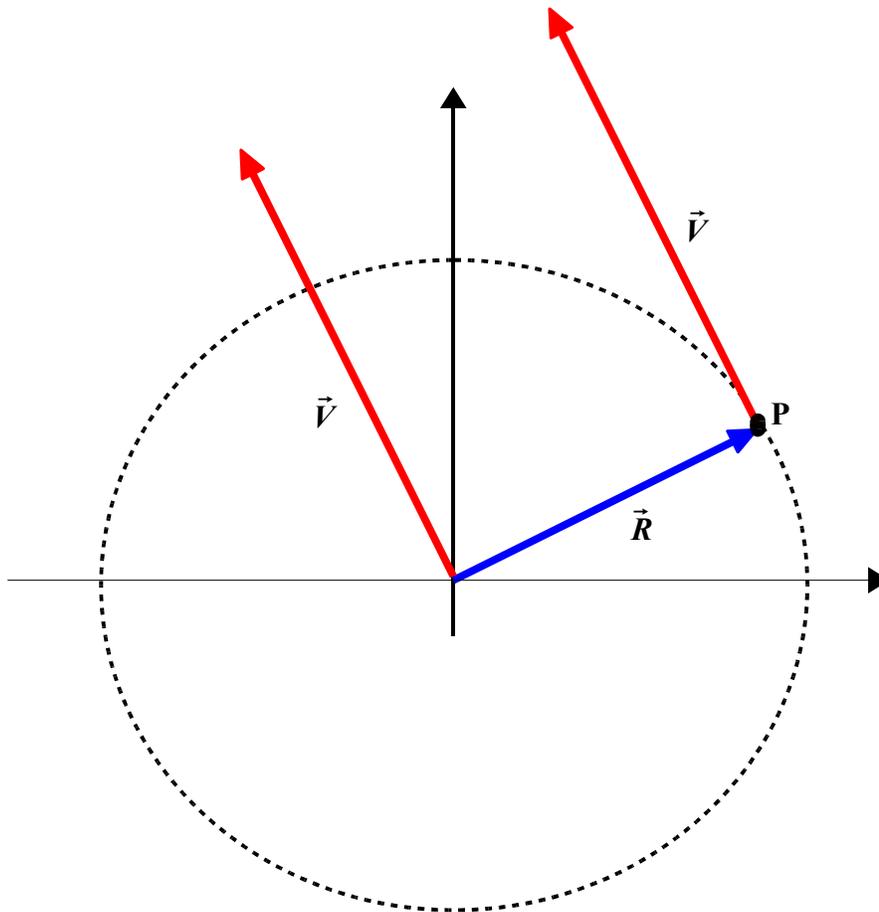
$\Phi$  radians. Thus  $\omega = \frac{d\Phi}{dt}$ . Let  $\vec{R}$  be the vector from the center to  $\mathbf{P}$ . We can represent the coordinates of  $\mathbf{P}$ , as  $(r \cos \Phi, r \sin \Phi)$ . This point specifies the location of the tip end of the vector  $\vec{F}$  which is moving around the circle. We imagine this vector rotating with its “tail” at the origin while its “head”, attached to  $\mathbf{P}$ , rotates counter-clockwise. We obtain the velocity vector  $\vec{V}$  by differentiating the coordinates of  $\mathbf{P}$  with respect to the time,  $t$ . Thus

$$\frac{d}{dt}(r \cos \Phi, r \sin \Phi) = \left( r(-\sin \Phi) \frac{d\Phi}{dt}, r(\cos \Phi) \frac{d\Phi}{dt} \right) = (-r \omega \sin \Phi, r \omega \cos \Phi)$$

We can associate vectors starting at the origin with their “head” coordinates. Thus we have

$$\begin{aligned} \vec{R} &= (r \cos \Phi, r \sin \Phi) \\ \vec{V} &= (-r \omega \sin \Phi, r \omega \cos \Phi) \end{aligned}$$

Let's pause and see what we have done here by graphing both of these vectors.



Notice that  $\vec{V}$  is drawn in two places and that it is perpendicular to  $\vec{R}$ . The slope of

$\vec{R}$  is  $\frac{r \sin \Phi}{r \cos \Phi} = \tan \Phi$ . Looking at  $\vec{V}$ , placed at the origin with its head at

$(-r\omega \sin \Phi, r\omega \cos \Phi)$ , we can get its slope as  $\frac{r\omega \cos \Phi}{-r\omega \sin \Phi} = -\cot \Phi = \frac{-1}{\tan \Phi}$

showing that  $\vec{V} \perp \vec{R}$ . We drew the velocity vector with its “toe” at the “head” of  $\vec{R}$  as it is an excellent way to show the direction of the movement of  $\mathbf{P}$  as well as the magnitude of the velocity as the length of  $\vec{V}$ . Note that the magnitude of  $\vec{V}$  is given by  $\|\vec{V}\| = \sqrt{(-r\omega \sin \Phi)^2 + (r\omega \cos \Phi)^2} = r\omega$ . This proves the very reasonable idea that the magnitude of the velocity of a rotating object along a circle of constant radius is proportional to its angular velocity.

Now let's consider the centrifugal force. We use Newton's Law,  $\mathbf{F} = m\mathbf{a}$ , (force equals mass times acceleration) to get the centrifugal force. First we obtain the acceleration vector,  $\vec{A}$ , by differentiating  $\vec{V} = (-r\omega \sin \Phi, r\omega \cos \Phi)$  as follows:

$$\begin{aligned}\vec{A} &= \frac{d}{dt}(\vec{V}) = \frac{d}{dt}(-r\omega \sin \Phi, r\omega \cos \Phi) \\ \vec{A} &= \left( -r\omega(\cos \Phi) \frac{d\Phi}{dt}, -r\omega(\sin \Phi) \frac{d\Phi}{dt} \right) = (-r\omega^2 \cos \Phi, -r\omega^2 \sin \Phi) \\ \vec{A} &= -\omega^2 \vec{R}\end{aligned}$$

This result, that the acceleration vector is  $-\omega^2$  times  $\vec{R}$  means that the acceleration is directed inward toward the center. The force vector, according to Newton's Law, can be written as

$$\vec{F} = m\vec{A} = -m\omega^2 \vec{R}$$

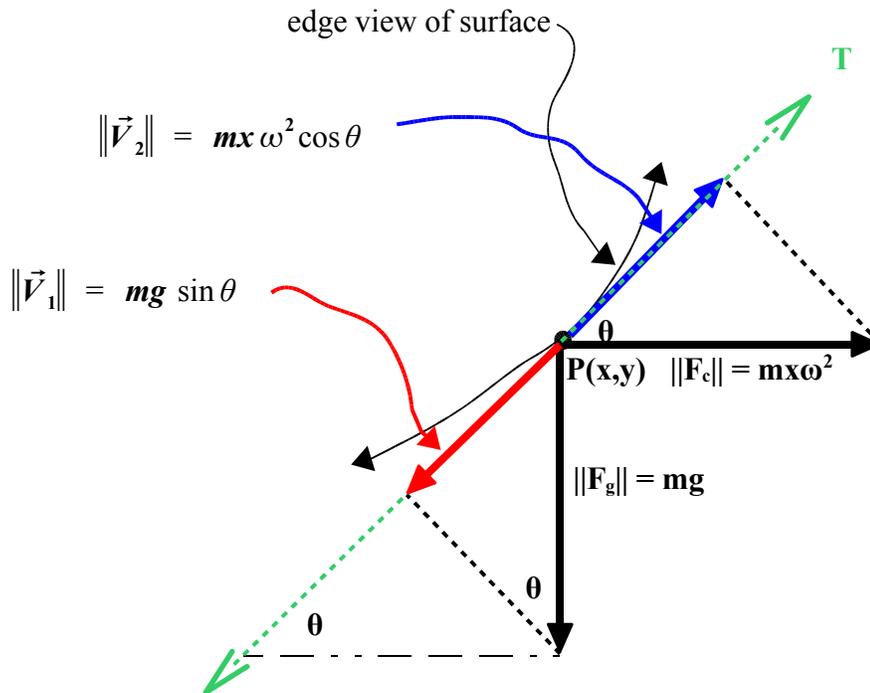
The best way to understand why the force is directed inward is to think of swinging a rock on the end of a string around in a circle. The string pulls the rock toward the center. The centrifugal force is the force the rock exerts on the string and is *opposite* to the force given by Newton's Law. The force statement above can be written as a magnitude statement

$$\|\vec{F}\| = m\omega^2(\|\vec{R}\|) = mr\omega^2$$

which has the added feature that we do not have to worry about any sign. The centrifugal force is thus proportional to the *square* of the angular velocity.

## Just Why Must the Surface be a Paraboloid?

To answer this question we now concentrate on the second view shown earlier, in particular on the tangent line to the curve of the side view of the surface. We draw a magnified version of the second view on the next page.



Study carefully the geometry of the diagram. Consider a small volume of liquid, mass  $m$ , centered at point  $P$ . While rotating at constant velocity, the gravity force and centrifugal forces are in balance, and it is this balance which shapes the surface of the liquid. Line  $T$  is tangent to the cut away side view of the surface at point  $P$ . ***It is in the direction of this tangent line that the force balance occurs.*** Let  $\theta$  be the angle of inclination of  $T$ . The vertical black vector is the gravity force vector and the red vector is the component of the force of gravity along the tangent line. This **vector**,  $\vec{V}_1$  extends from  $P$  to the foot of the perpendicular dropped to  $T$  from the “head” of the gravity force vector. By similar triangles,  $\theta$  appears opposite the red vector, so the length (magnitude) of  $\vec{V}_1$  is  $mg \sin \theta$  by using right triangle trigonometry, since the hypotenuse has length  $mg$ . The horizontal black vector is the centrifugal force vector which has the magnitude  $m\omega^2 x$  as seen earlier. It forms the hypotenuse of a triangle formed by dropping a perpendicular from its “head” to line  $T$ . The blue vector,  $\vec{V}_2$ , is the component of the centrifugal force along the tangent line, and by right triangle trigonometry, its magnitude is  $m\omega^2 x \cos \theta$ .

The balance of forces requires that

$$\begin{aligned} \|\vec{V}_1\| &= \|\vec{V}_2\| \Rightarrow mg \sin \theta = m\omega^2 x \cos \theta \\ &\Rightarrow \frac{\omega^2}{g} x = \tan \theta \end{aligned}$$

Since the tangent of the inclination angle of a line equals the slope of that line,  $\tan \theta$  is also the slope of the curve through  $P(x, y)$  which is the derivative of the curve at  $p$ . Thus we have the differential equation

$$\frac{dy}{dx} = \frac{\omega^2}{g} x$$

Separating the variables and integrating, we obtain

$$\int dy = \frac{\omega^2}{g} \int x dx \Rightarrow y = \left( \frac{\omega^2}{2g} \right) x^2 + C$$

Assuming the curve contains the origin at its lowest point (the coordinate system's placement is arbitrary) we have  $x = 0$  when  $y = 0$ . This makes  $C = 0$  so the curve of the edge view of the surface is the ***parabola***

$$y = \left( \frac{\omega^2}{2g} \right) x^2$$

proving that the surface is a ***paraboloid!***