

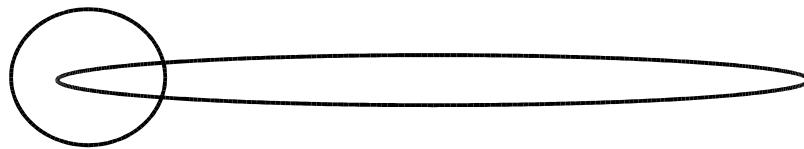
# The Eccentricity Story

## Introduction

The concept of eccentricity, like the general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

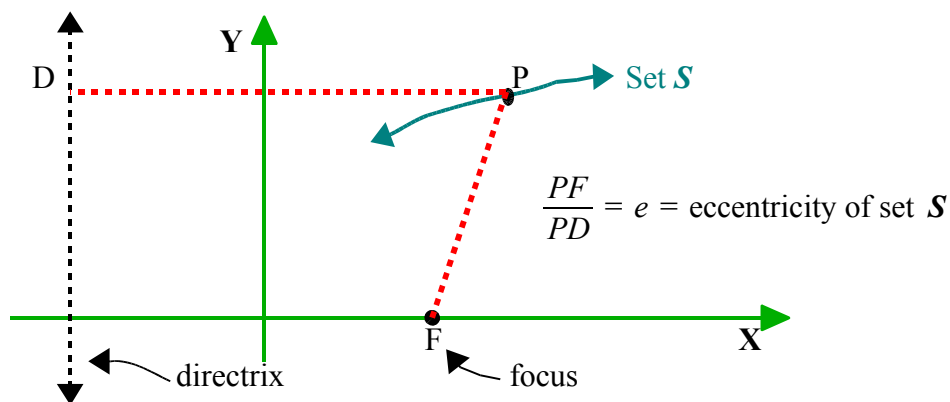
is a unifying concept for the conic sections: circle, ellipse, parabola, and hyperbola. One of the greatest uses of eccentricity is in astronomy. As we shall see, the circle has an eccentricity of zero while the eccentricity of an ellipse varies from 0 to 1, not inclusively. As the eccentricity of an ellipse approaches 1, it becomes more highly elongated. Planetary orbits all have eccentricities between 0 and 1. The eccentricity of the earth is approximately 0.0166666, indicating a very nearly circular orbit. At the other extreme is the eccentricity of the orbit of comet Kouhoutek which is 0.99993. This means it's orbit is an exceedingly elongated ellipse which requires the comet 200,000 years to complete!

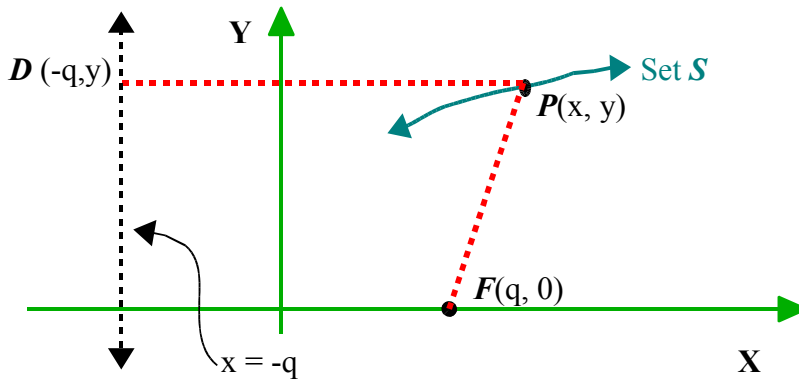


The figure above shows the relative scale of Kouhoutek's orbit, the ellipse, with the size of the solar system, the region inside the circle.

## What is Eccentricity?

**Definition:** Let a fixed point  $F$ , called the focus, and a fixed line,  $L$ , called the directrix be given. Let  $S$  be the set of points such that given any point  $P$  in set  $S$ , the **ratio** formed by the distance from  $P$  to  $F$  divided by the distance from  $P$  to  $L$  **is constant**. The value of this **ratio** is called the eccentricity of set  $S$ . The diagram below illustrates this definition.





We will now apply the definition to the situation shown in the diagram above in which for convenience we have placed the focus at  $F(q, 0)$ , and the directrix,  $x = -q$ , on opposite sides of the origin. Let point  $P$  be placed at  $(x, y)$ .

### Applying the Definition

According to the definition of the eccentricity,  $e$ , of the set of points  $P$  we require that  $\frac{PF}{PD} = e$ . Using the distance formula we proceed to apply the definition to obtain an equation for set  $S$ .

$$\begin{aligned} PF &= e(PD) \\ \sqrt{(x - q)^2 + y^2} &= e\sqrt{(x + q)^2 + (y - y)^2} \\ (x - q)^2 + y^2 &= e^2(x + q)^2 \end{aligned}$$

After squaring out and rearranging terms we get

$$y^2 + (1 - e^2)x^2 - 2q(1 + e^2)x = q^2(e^2 - 1) \quad (1)$$

### $e = 0$

If we let  $e = 0$ , equation (1) becomes

$$\begin{aligned} y^2 + x^2 - 2qx + q^2 &= 0 \\ y^2 + (x - q)^2 &= 0 \end{aligned}$$

The graph of this equation is the single point  $(q, 0)$ .

### $1 > e > 0$

Now suppose  $e = 1$ . Substituting 1 for  $e$  in equation (1) results in

$$y^2 = 4qx$$

which, no doubt, you recognize as a standard form parabola with its vertex at the origin.

Now we assume  $0 < e < 1$  and return to equation (1). Let's try to write it in the general form of an ellipse which you should remember is the form  $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ .

We first factor out  $(1-e^2)$  from the two x-terms in preparation for the completing the square maneuver.

$$y^2 + (1-e^2) \left\{ x^2 - \frac{2q(1+e^2)}{1-e^2}x + \dots \right\} = q^2(e^2-1)$$

Next, we add the square of half of the coefficient of x inside the braces in place of the ellipsis. To maintain the equation's balance we add  $(1-e^2)$  times this quantity to the right hand side.

$$y^2 + (1-e^2) \left\{ x^2 - \frac{2q(1+e^2)}{1-e^2}x + \frac{q^2(1+e^2)^2}{(1-e^2)^2} \right\} = q^2(e^2-1) + \frac{q^2(1+e^2)^2}{1-e^2}$$

We factor and dress up the right side.

$$\left( x - \frac{q(1+e^2)}{1-e^2} \right)^2 (1-e^2) + y^2 = \frac{-q^2(e^2-1)^2 + q^2(1+e^2)^2}{1-e^2}$$

Dividing both sides by  $(1-e^2)$  and simplifying the numerator on the right side, the following results.

$$\begin{aligned} \left( x - \frac{q(1+e^2)}{1-e^2} \right)^2 + \frac{y^2}{1-e^2} &= \frac{q^2 \{ (e^4 - 2e^2 + 1)(-1) + 1 + 2e^2 + e^4 \}}{(1-e^2)^2} \\ \left( x - \frac{q(1+e^2)}{1-e^2} \right)^2 + \frac{y^2}{1-e^2} &= \frac{4q^2e^2}{(1-e^2)^2} \end{aligned}$$

At this point we are ready to divide both sides by  $\frac{4q^2e^2}{(1-e^2)^2}$  in order to get 1 on the right side. The result is

$$\frac{\left(x - \frac{q(1+e^2)}{1-e^2}\right)^2}{\left(\frac{2qe}{1-e^2}\right)^2} + \frac{y^2}{\left(\frac{4q^2e^2}{1-e^2}\right)} = 1 \quad (2)$$

Compare this with  $\frac{(x-h)^2}{a^2} + \frac{(y-0)^2}{b^2} = 1$ . Clearly  $a = \frac{2qe}{1-e^2}$ , and

$b^2 = \frac{4q^2e^2}{1-e^2}$  makes sense only if  $(1-e^2) > 0$  as  $b^2$  is never negative. Therefore we require that  $1 > e^2$  which means that  $0 < e < 1$ . Therefore equation (2) describes an ellipse whose eccentricity lies between 0 and 1.

## **e > 1**

Let's take equation (2) and sneak a minus sign between the two terms on the left side, legally, of course with one other alteration done to the first term.

$$\frac{\left(x - \frac{q(1+e^2)}{1-e^2}\right)^2}{\left(\frac{2qe}{e^2-1}\right)^2} - \frac{y^2}{\left(\frac{4q^2e^2}{e^2-1}\right)} = 1 \quad (3)$$

Compare (3) with the standard form  $\frac{(x-h)^2}{a^2} - \frac{(y-0)^2}{b^2} = 1$ . We have

$a = \frac{2qe}{e^2-1}$ , and  $b^2 = \left(\frac{4q^2e^2}{e^2-1}\right)$  with both  $a$  being positive and  $b^2$  being positive if  $e^2 - 1 > 0$ . In this case  $e^2 > 0$  or  $e > 1$  causes (3) to be a hyperbola.

## **Summary**

We summarize the results in the table.

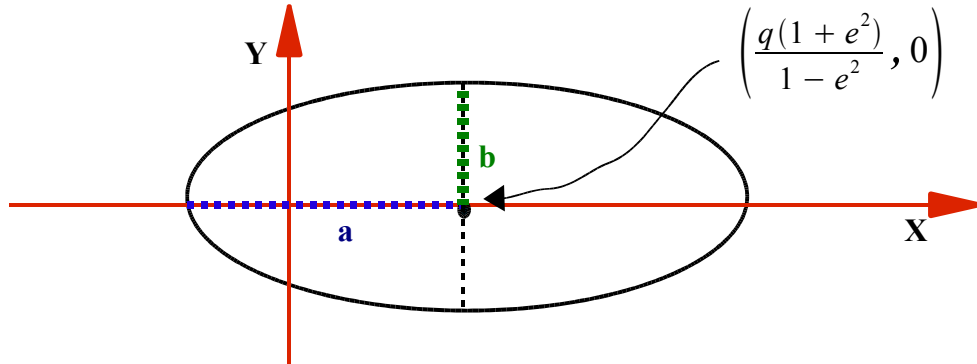
<i>Conic Section</i>	<i>Eccentricity</i>
Ellipse	$0 < e < 1$
Parabola	$1 = e$
Hyperbola	$1 < e$

## What about the Circle?

We discuss the circle eccentricity by considering the ellipse. Look again at equation (2) which we repeat below in slightly altered form.

$$\frac{\left(x - \frac{q(1+e^2)}{1-e^2}\right)^2}{\left(\frac{2qe}{1-e^2}\right)^2} + \frac{y^2}{\left(\frac{2qe}{\sqrt{1-e^2}}\right)^2} = 1$$

We have  $a = \frac{2qe}{1-e^2}$  and  $b = \frac{2qe}{\sqrt{1-e^2}}$ . The graph is shown below.

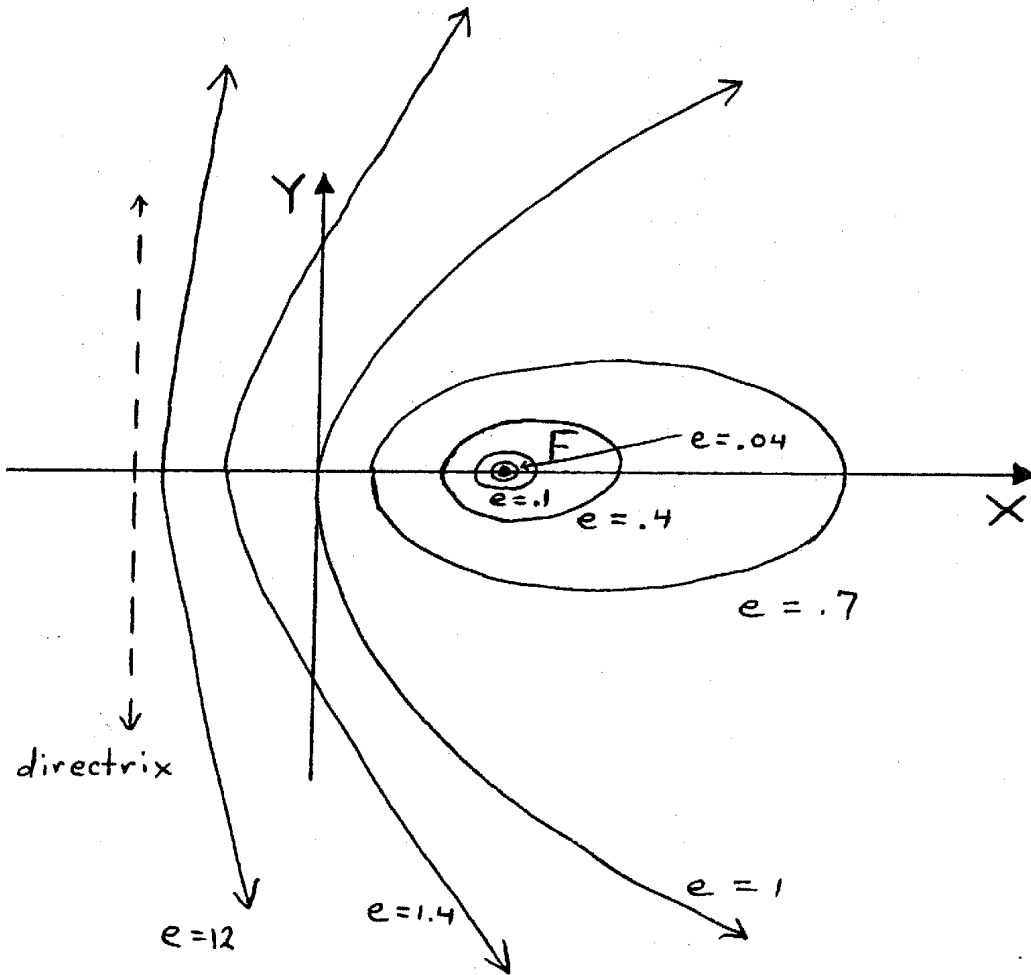


If we let  $e$  approach 0, the ratio of  $b$  to  $a$  approaches 1. To see why, consider the following.

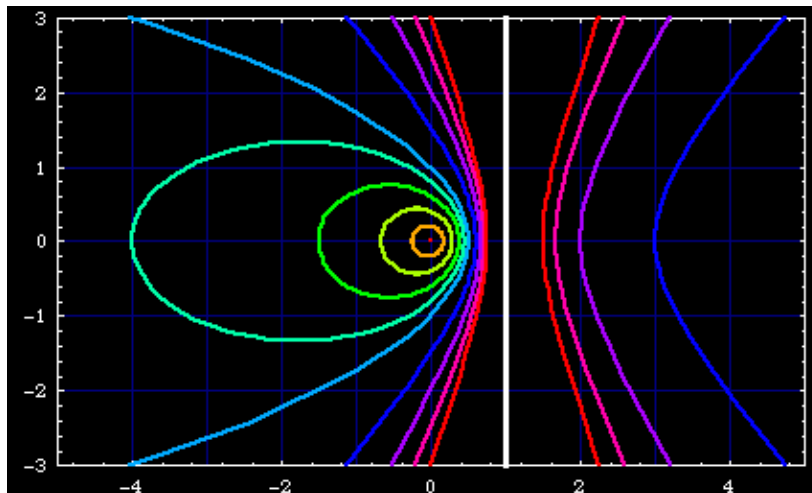
$$\frac{b}{a} = \frac{\left(\frac{2qe}{\sqrt{1-e^2}}\right)}{\left(\frac{2qe}{1-e^2}\right)} = \frac{1-e^2}{\sqrt{1-e^2}} = \sqrt{1-e^2}$$

As  $e \rightarrow 0$ ,  $\frac{b}{a} \rightarrow 1$  so that the size of  $b$  approaches the size of  $a$ , becoming equal to the radius of a circle. Thus the circle has eccentricity zero.

The image on the next page shows seven graphs on the same coordinate system sharing the same focus and directrix. Eccentricities range from 0.04, 0.1, 0.4, 0.7, 1, 1.4, and 12. The second branch of the hyperbolas with eccentricities 1.4 and 12 are not shown.

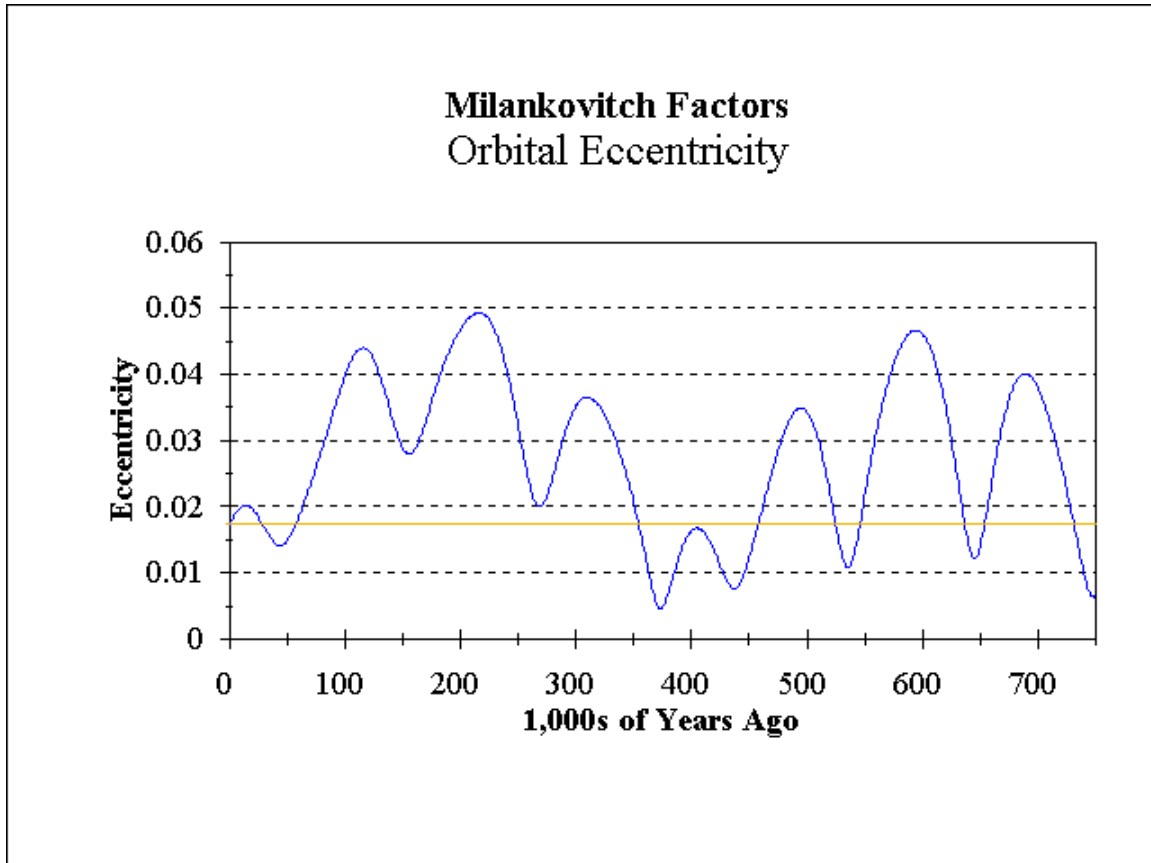


The graph below shows both branches of 4 hyperbolas, all graphs with the same focus and directrix.

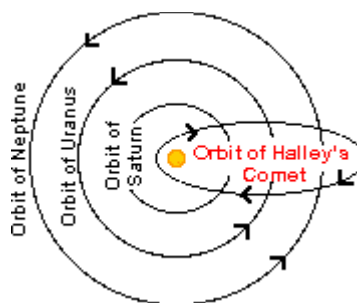


## Assorted Images of Interest

The eccentricity of the earth's orbit actually changes over time as shown in the graph below. Note the rhythmic pattern. Such changes, although small, do affect world temperatures.



Haley's Comet is the most famous comet. Its orbit, shown relative to some of the planets is shown in the next image.



All planetary orbits are conic sections with the sun at one focus. If a body comes close to the sun but travels at the critical speed, below which it would be captured, its path is a parabola. If moving a little faster, its path is one branch of a hyperbola. The last image

(yes, this has to end) shows these scenarios. New comets are carefully observed to determine if they have ever made a previous pass around the sun. Obviously if the path is a portion of a parabola or hyperbola, the comet is confirmed as being truly new.

