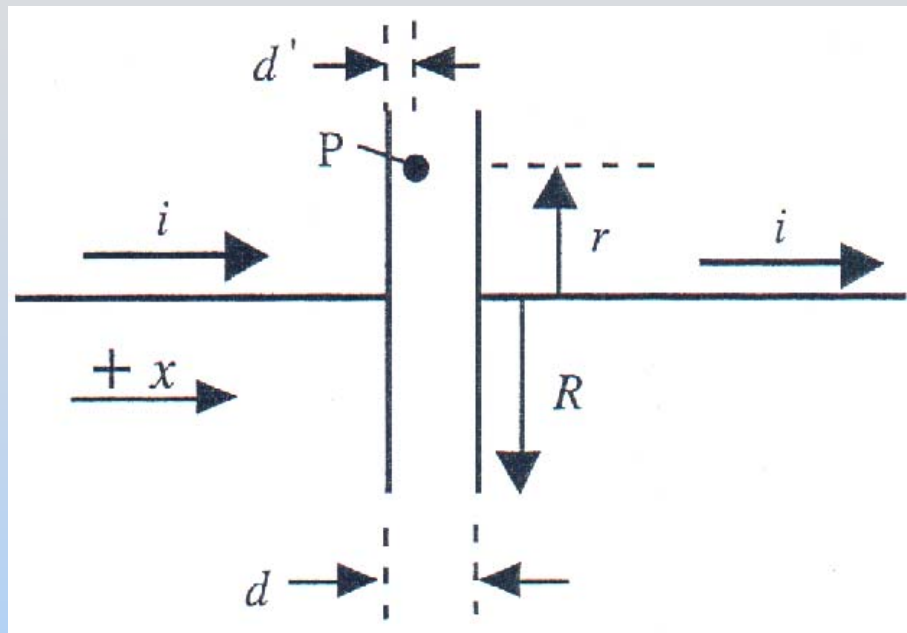


F2002 #5, S2003 #3, SFB#1, SFC#1, SFD#1

Problem 1: Gauss's Law

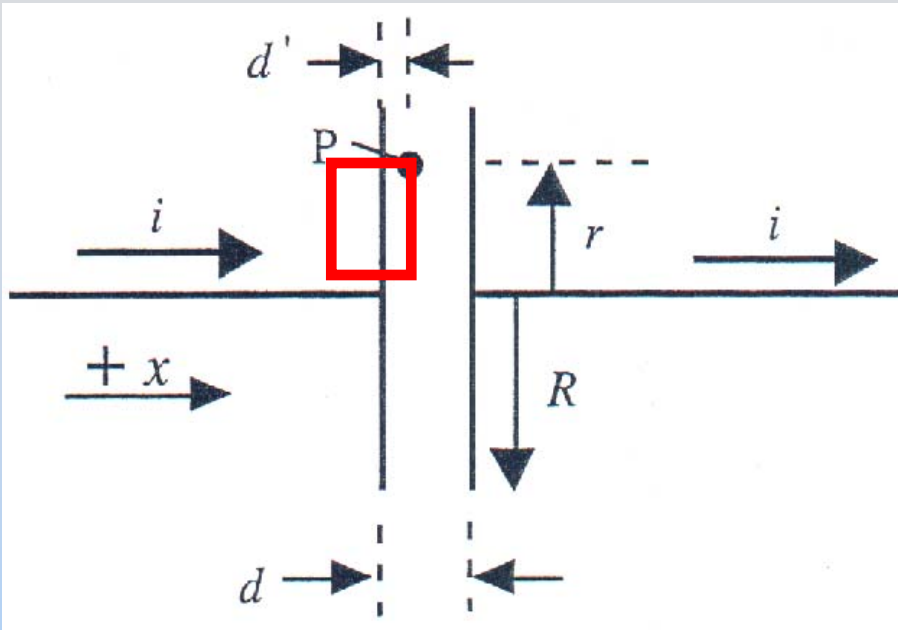


A circular capacitor of spacing d and radius R is in a circuit carrying the steady current i shown.

At time $t=0$ it is uncharged

1. Find the electric field $\mathbf{E}(t)$ at P vs. time t (mag. & dir.)
2. Find the potential at P , $V(t)$, given that the potential at the right hand plate is fixed at 0
3. Find the magnetic field $\mathbf{B}(t)$ at P
4. Find the total field energy between the plates $U(t)$

Solution 1: Gauss's Law



1. Find the electric field $\mathbf{E}(t)$:
Assume a charge q on the left plate ($-q$ on the right)

Gauss's Law:

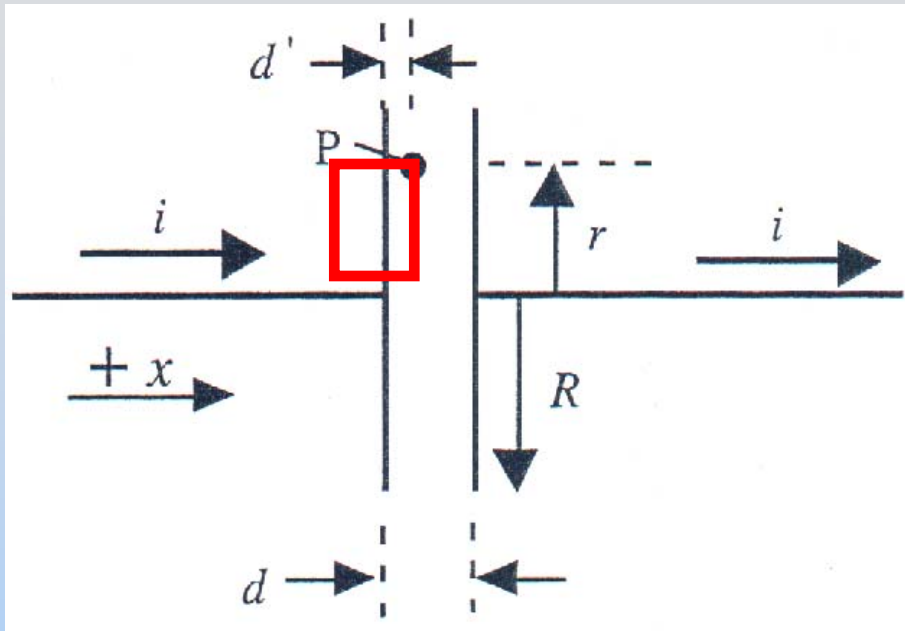
$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = \frac{Q_{in}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$E = \frac{\sigma}{\epsilon_0} = \frac{q}{\pi R^2 \epsilon_0}$$

Since $q(t=0) = 0$, $q = it$

$$\vec{\mathbf{E}}(t) = \frac{it}{\pi R^2 \epsilon_0} \text{ to the right}$$

Solution 1.2: Gauss's Law



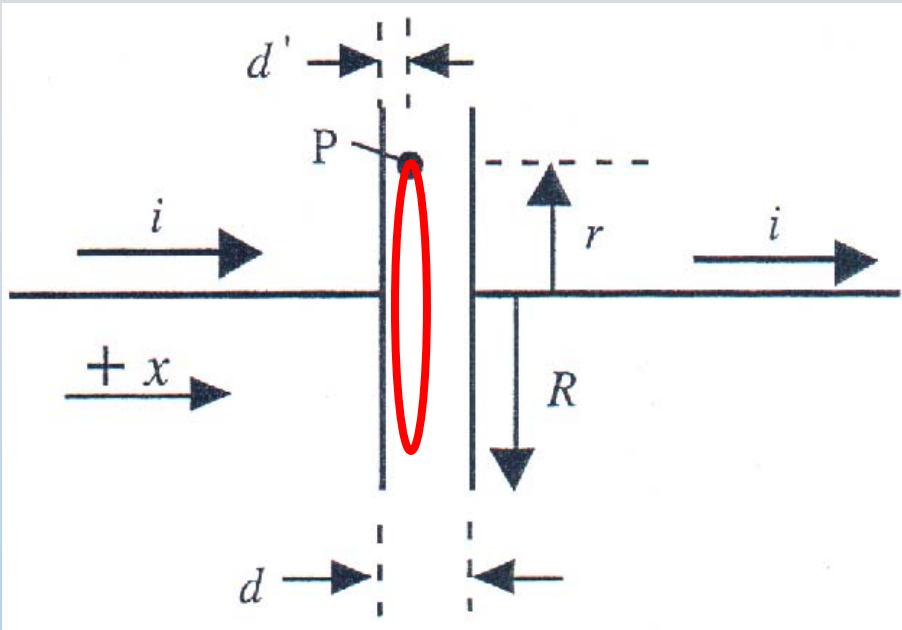
2. Find the potential $V(t)$:

Since the E field is uniform,
 $V = E * \text{distance}$

$$V(t) = |\vec{E}(t)| (d - d') = \frac{it}{\pi R^2 \epsilon_0} (d - d')$$

Check: This should be positive since its between a positive plate (left) and zero potential (right)

Solution 1.3: Gauss's Law



3. Find $\mathbf{B}(t)$:

Ampere's Law:

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

$$2\pi r B = 0 + \mu_0 \epsilon_0 \frac{r^2 i}{R^2 \epsilon_0}$$

$$\Phi_E = EA = \left(\frac{it}{\pi R^2 \epsilon_0} \right) \pi r^2$$

$$\frac{d\Phi_E}{dt} = \frac{r^2 i}{R^2 \epsilon_0}$$

$$\vec{\mathbf{B}}(t) = \frac{\mu_0 i r}{2\pi R^2} \text{ out of the page}$$

Solution 1.4: Gauss's Law

4. Find **Total Field Energy** between the plates

$$\text{E Field Energy Density: } u_E = \frac{\epsilon_0 E^2}{2} = \frac{\epsilon_0}{2} \left(\frac{it}{\pi R^2 \epsilon_0} \right)^2$$

$$\text{B Field Energy Density: } u_B = \frac{B^2}{2\mu_0} = \frac{1}{2\mu_0} \left(\frac{\mu_0 ir}{2\pi R^2} \right)^2$$

$$\text{Total Energy } U = \iiint (u_E + u_B) dV \quad (\text{Integrate over cylinder})$$

$$= \frac{\epsilon_0}{2} \left(\frac{it}{\pi R^2 \epsilon_0} \right)^2 \cdot \pi R^2 d + \frac{1}{2\mu_0} \left(\frac{\mu_0 i}{2\pi R^2} \right)^2 \int r^2 \cdot d \cdot 2\pi r dr$$

$$= \frac{(it)^2}{2} \frac{d}{\epsilon_0 \pi R^2} + \frac{1}{2} \frac{\mu_0 d}{8\pi} i^2 \quad \left(= \frac{q^2}{2C} + \frac{1}{2} Li^2 \right)$$