

## Maxwell's equations so far (2)

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = 4\pi\rho \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} \end{cases}$$

- Is this set of equations consistent? Not quite...

- Take the divergence of Ampere's law

- $\vec{\nabla} \cdot \left( \frac{4\pi}{c} \vec{J} \right) = \frac{4\pi}{c} \vec{\nabla} \cdot \vec{J} = -\frac{4\pi}{c} \frac{\partial \rho}{\partial t}$  (using continuity equation)

- $\vec{\nabla} \cdot \vec{\nabla} \times \vec{B} = 0$  ( $\vec{\nabla} \cdot \vec{\nabla} \times \vec{v}$  is ALWAYS 0!)

Ampere's law works only when  $d\rho/dt=0$  which works in most cases but not always: Ampere's law is incomplete!

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## Fixing the inconsistency

- Since  $\vec{\nabla} \cdot \vec{\nabla} \times \vec{v} \equiv 0$  we need to add some term to the right hand side to that its divergence will be identically 0

- Generalized Ampere's law:  $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \vec{F}$

- What is F? We know that its divergence must be =0:

$$\vec{\nabla} \cdot \left( \frac{4\pi}{c} \vec{J} + \vec{F} \right) = 0 \Rightarrow \vec{\nabla} \cdot (c\vec{F}) = -4\pi \vec{\nabla} \cdot \vec{J} = 4\pi \frac{\partial \rho}{\partial t}$$

$$\Rightarrow \vec{\nabla} \cdot (c\vec{F}) = 4\pi \frac{\partial \rho}{\partial t} \quad \text{Similar to Gauss's law!}$$

- Take time derivative of Gauss's law:

$$\frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E}) = 4\pi \frac{\partial \rho}{\partial t} \Rightarrow \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E}) = \vec{\nabla} \cdot \left( \frac{\partial \vec{E}}{\partial t} \right) = \vec{\nabla} \cdot (c\vec{F}) \quad \text{time and space derivatives commute}$$

$$\Rightarrow \vec{F} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

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# Displacement currents

- Generalized Ampere's equation  $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$
- This can also be written as:  $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} (\vec{J} + \vec{J}_d)$ 
  - With  $J_d =$  displacement current (density):  $\vec{J}_d = \frac{1}{4\pi} \frac{\partial \vec{E}}{\partial t}$
- What is the  $J_d$ ?
  - Not a real current: does not describe charges flowing through some region
  - But it acts like a real current: whenever we have changing E field, we can treat its effect as if due to as a real current  $J_d$

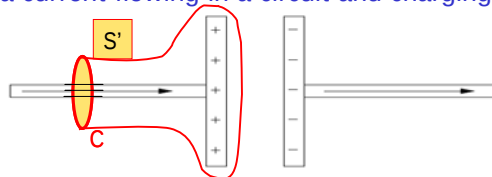
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# What is a displacement current?

- Consider a current flowing in a circuit and charging a capacitor C



- Standard integral Ampere's law:  $\oint_C \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I_{encl} = \frac{4\pi}{c} \int_S \vec{J} \cdot d\vec{a}$ 
  - Let's choose the path C and the surface S as in the drawing above:
    - It all makes sense!
  - Now choose the same path C but the surface S' (ok by Stokes...)
    - No standard current J through the surface (no charge crosses C!)
    - But there is a flux of displacement current  $J_d$  through the plates!

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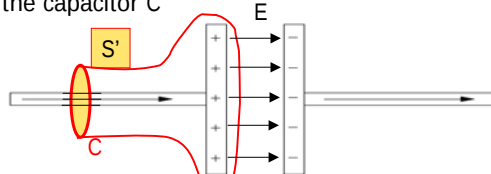
## What is a displacement current? (2)

- We can use the generalized Ampere's Law:

$$\oint_C \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} (I_{encl} + I_d)$$

$$\text{with } I_d = \int_S \vec{J}_d \cdot d\vec{a} = \frac{1}{4\pi} \int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \frac{1}{4\pi} \frac{\partial}{\partial t} \int_S \vec{E} \cdot d\vec{a} = \frac{1}{4\pi} \frac{\partial \Phi_{\vec{E}}}{\partial t}$$

- The displacement current is related to the change over time of the flux of the electric field.
  - In the example above, the electric field  $E$  is the one produced in between the plates of the capacitor  $C$



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## What is a displacement current? (3)

- The electric field  $E$ :
  - Points in the same direction of the current (+x)
  - At a given instant in time:  $\vec{E} = \frac{4\pi Q}{A} \hat{x}$
- The flux of  $E$  will then be:  $\Phi_{\vec{E}} = 4\pi Q$  (yes, Gauss's law!)
- The rate of the change of this flux is:  $\frac{\partial \Phi_{\vec{E}}}{\partial t} = 4\pi \frac{\partial Q}{\partial t} = 4\pi I$ 
  - Where  $I$  is the current that is charging the capacitor
- Comparing this with results in the previous page:

$$I_d = \int_S \vec{J} \cdot d\vec{a} = \int_S \vec{J}_d \cdot d\vec{a} = I$$

→ generalized Ampere's Law is valid no matter what surface we use

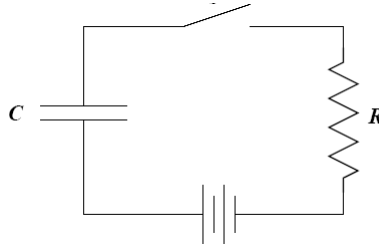
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## The importance of displacement currents

- When we examined the following circuit:



we said the same current  $I$  was flowing in each circuit element.

- How is it possible? No current flows through the plates of a capacitor!
  - Displacement currents fix this inconsistency!
  - Displacement current “continues” the “real” current across the capacitors ensuring the validity of Kirchoff’s laws.

## Maxwell’s equations (complete!)

|                                                                                                                                                                                                                                                                                                                                         |                                                                                         |                                                                                                                                                                                                                                                                                                                   |                                                                                        |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------|
| $\left\{ \begin{array}{l} \vec{\nabla} \cdot \vec{E} = 4\pi\rho \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \end{array} \right.$                      | <div style="background-color: yellow; padding: 2px; border: 1px solid black;">cgs</div> | $\left\{ \begin{array}{l} \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{array} \right.$ | <div style="background-color: yellow; padding: 2px; border: 1px solid black;">SI</div> |
| <div style="display: flex; justify-content: space-around; align-items: center;"> <div style="text-align: center;"> <math>\nwarrow</math><br/> <div style="background-color: orange; padding: 2px; border: 1px solid black;">Generalized Ampere's law</div> </div> <div style="text-align: center;"> <math>\nearrow</math> </div> </div> |                                                                                         |                                                                                                                                                                                                                                                                                                                   |                                                                                        |

Typo in Purcell  
Eq 15 ch 9

NB: when Maxwell introduced the term  $d\vec{E}/dt$  in the generalized Ampere's law, his arguments were based purely on symmetry

- Yes, he was a theorist!

## Maxwell's equations: integral form

$$\left\{ \begin{array}{ll} \Phi_{\vec{E}} = \int_S \vec{E} \cdot d\vec{a} = 4\pi Q_{enc} & \text{(Gauss's law)} \\ \Phi_{\vec{B}} = 0 & \text{(Magnetic field line are closed)} \\ emf = \oint_C \vec{E} \cdot d\vec{l} = -\frac{1}{c} \frac{\partial \Phi_{\vec{B}}}{\partial t} & \text{(Faraday's law)} \\ \oint_C \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} (\vec{I} + \vec{I}_d) & \text{(Generalized Ampere's law)} \end{array} \right.$$

where the currents  $\vec{I}$  and  $\vec{I}_d$  are defined as  $\vec{I} = \int_S \vec{J} \cdot d\vec{a}$  and

$$\vec{I}_d = \frac{1}{4\pi} \frac{\partial \Phi_{\vec{E}}(S)}{\partial t}$$

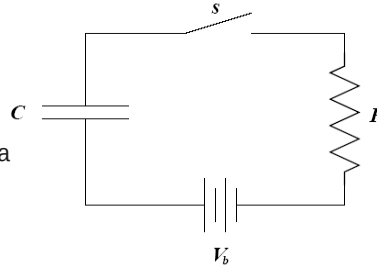
## 3 good reasons to remember Maxwell's equations

- 1) They compactly and beautifully summarize all the E&M we learned so far!
- 2) You will see them on T shirts for the rest of your life at MIT:  
better to get familiar with them ASAP!
- 3) On the first day of 8.03 next semester you will be asked to write them down on a piece of paper to check what you learned in your first semester at MIT: save your honor (and mine)

## Displacement current: application

- Consider the following RC circuit:

- As C charges up,  $I_d$  flows
- $I_d$  induces B inside the plates
- Assuming cylindrical plates of radius a



- Calculate B inside the plates

1) Find  $\vec{E}(t)$ :  $\vec{E}(t) = 4\pi\sigma = \frac{4\pi Q(t)}{\pi a^2}$

2) Displ. current density:  $\vec{J}_d = \frac{1}{4\pi} \frac{\partial \vec{E}}{\partial t} = \frac{1}{4\pi} \frac{\partial E(t)}{\partial t} = \frac{1}{\pi a^2} \frac{\partial Q(t)}{\partial t} = \frac{I(t)}{\pi a^2}$

3) Remember that  $I(t) = \frac{V_b}{R} e^{-t/RC}$

4) Magnetic field inside the plate (Ampere's law):  $\oint_C \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} \int_C \vec{J}_d \cdot d\vec{a}$

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$$\Rightarrow B(r) = \frac{2rV_b}{ca^2R} e^{-t/RC} \quad 13$$

## Maxwell equations in vacuum

- What happens when we write Maxwell's equations in vacuum?

- Vacuum: no sources,  $\rho=0$  and  $J=0$

$$\left\{ \begin{array}{l} \vec{\nabla} \cdot \vec{E} = 4\pi\rho \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \end{array} \right. \rightarrow \left\{ \begin{array}{l} \vec{\nabla} \cdot \vec{E} = 0 \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \end{array} \right.$$

- Except for a – sign, these equations are exquisitely symmetric!
- Consequence: an electric field  $E$  varying in time will create a magnetic field  $B$ ; a  $B$  varying in time creates a  $E$ :  $E$  and  $B$  are intimately related!

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## Maxwell equations in vacuum: solution

- How to solve these equations?

- Uncouple them!

- Separate E and B in equations

- How?

- Take the curl of equations (3) and (4)
- Use other equations as needed

$$\vec{\nabla} \cdot \vec{E} = 0 \quad (1)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad (3)$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \quad (4)$$

- Start from (3):

$$\text{Left: } \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = -\vec{\nabla}^2 \vec{E} \quad (\text{since } \vec{\nabla} \cdot \vec{E} = 0 \text{ in vacuum})$$

$$\text{Right: } \vec{\nabla} \times \left( -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \right) = -\frac{1}{c} \frac{\partial}{\partial t} \vec{\nabla} \times \vec{B} = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \quad (\text{using (4)})$$

$$\Rightarrow \boxed{\vec{\nabla}^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}}$$

## Maxwell equations in vacuum: solution

- Now repeat the procedure starting from  $\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \quad (4)$

$$\text{Left: } \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{B}) - \vec{\nabla}^2 \vec{B} = -\vec{\nabla}^2 \vec{B} \quad (\text{since } \vec{\nabla} \cdot \vec{B} = 0 \text{ in vacuum})$$

$$\text{Right: } \vec{\nabla} \times \left( \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right) = \frac{1}{c} \frac{\partial}{\partial t} \vec{\nabla} \times \vec{E} = -\frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} \quad (\text{using (3)})$$

$$\Rightarrow \boxed{\vec{\nabla}^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}}$$

- This is a special case of a known equation: the wave equation:

$$\boxed{\vec{\nabla}^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}} \quad \text{where } f = f(x \pm vt)$$

where f is any function that has well-behaved derivatives

NB: we are restricting ourselves to the 1D case; extension to 3D next lecture

$$\vec{\nabla}^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

## Solution of wave equation: prove

- Prove that  $f = f(x \pm vt)$  is a solution of the wave equation
- Just calculate time and space derivatives.
  - Keep in mind that  $\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
- Define  $u = x \pm vt$

$$\frac{\partial f(x \pm vt)}{\partial t} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial t} = \pm v \frac{\partial f}{\partial u} \Rightarrow \frac{\partial^2 f(x \pm vt)}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial u^2}$$

$$\frac{\partial f(x \pm vt)}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} = \frac{\partial f}{\partial u} \Rightarrow \frac{\partial^2 f(x \pm vt)}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}$$

$$\text{Plug the above results into the equation} \Rightarrow \frac{\partial^2 f}{\partial u^2} = \frac{1}{v^2} v^2 \frac{\partial^2 f}{\partial u^2} \Rightarrow \text{identity!}$$

- As we wanted to prove!

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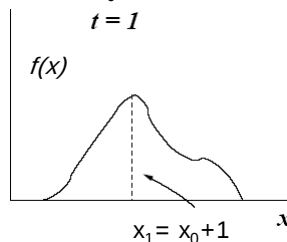
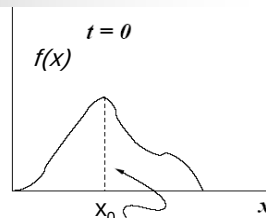
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$$\vec{\nabla}^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

## Wave equation solution

- What is a function such as  $f = f(x - vt)$  ?
  - Assume  $v=1$  cm/s
- At time  $t=0$ :
  - Position of the max:  $x_0$
- At time  $t=1$  s:
  - The peak still occurs when the argument of  $f$  is  $x_0$
  - But since the time is not 0  
→ the function will be shifted in  $x$  by " $vt$ "=1 cm
  - Position of the max:  $x_1 = x_0 + 1$



$f = f(x - vt)$  represents a wave traveling in the  $+x$  direction with velocity  $v$

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$$\bar{\nabla}^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

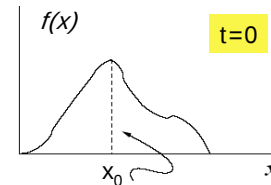
## Wave equation solution

- What is a function such as  $f = f(x + vt)$  ?

- Assume  $v=1$  cm/s

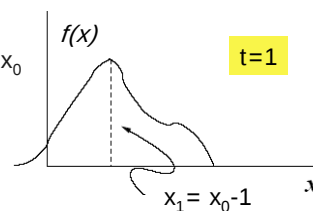
- At time  $t=0$ :

- Position of the max:  $x_0$



- At time  $t=1$  s:

- The peak still occurs when the argument of  $f$  is  $x_0$
- But since the time is not 0  
→ the function will be shifted in  $x$  by " $vt$ "=1 cm
- Position of the max:  $x_1 = x_0 - 1$



$f = f(x + vt)$  represents a wave traveling in the  $-x$  direction with velocity  $v$

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## EM waves

- Wave equation:  $\bar{\nabla}^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$

- Solution:  $f = f(x \pm vt)$

- Any function of argument  $x \pm vt$
- These solution represent waves traveling with velocity  $v$ 
  - $x - vt$  represents a wave traveling in the  $+x$  direction
  - $x + vt$  represents a wave traveling in the  $-x$  direction

- Maxwell's equation:  $\bar{\nabla}^2 f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$

- Same equation! Only difference:  $v=c$
- Solution: EM waves traveling with speed of light  
→ The light IS an EM wave!!!

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## EM waves in SI

- This same result looks much more interesting in SI.
- Maxwell's equations in SI:

$$\vec{\nabla}^2 f = \mu_0 \epsilon_0 \frac{\partial^2 f}{\partial t^2}$$

where  $\epsilon_0$  is the permittivity of free space  
and  $\mu_0$  is the permeability of free space

- Maxwell's equations tell us what the velocity of an EM wave is:

$$v = 1 / \sqrt{\mu_0 \epsilon_0}$$

- $\epsilon_0$  and  $\mu_0$  can be measured → we can predict velocity of EM waves:

$$\begin{aligned}\epsilon_0 &= 8.85418 \times 10^{-12} \text{ Coulomb}^2 \text{ Newton}^{-1} \text{ meter}^{-2} \\ \mu_0 &= 4\pi \times 10^{-7} \text{ Newton sec}^2 \text{ Coulomb}^{-2}\end{aligned}$$

→  $v = 2.998 \times 10^8 \text{ m/s}$  which is the speed of light!

- Maxwell was the first to realize that E&M equations were leading to a wave equation that was propagating at the speed of light: **light is an EM wave!**

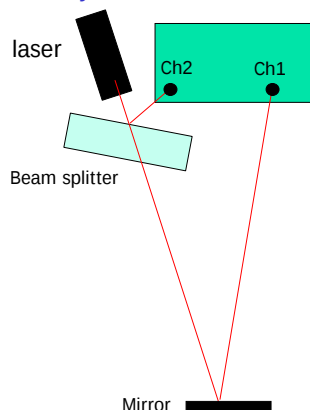
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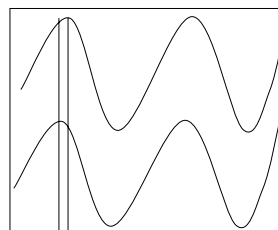
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## How to measure c (demo A4)

- Experimental setup: a neon laser beam is sent into a beam splitter. Part of it is reflected and part of it is refracted first and then reflected by a mirror.



- Difference in path between the 2 beams:  
~  $17.15 \text{ m} \times 2 = 34.3 \text{ meters}$
  - Measure the delay of channel 2 wrt channel 1 on the scope: 116 ns
- $v = 34.3 \text{ m} / 116 \text{ ns} = 2.96 \times 10^8 \text{ m/s}$



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# Summary and outlook

## ■ Today:

- Complete Maxwell's equations
  - The missing term leads to displacement currents
- Solution of Maxwell's equations in vacuum
  - Wave equation  $\rightarrow$  light is an EM radiation

## ■ Next time:

- Properties of EM radiation
- Polarization and scattering of light