

(ii) For the semi-circular arc of radius r , we make use of the Biot-Savart law:

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} \quad (9.11.17)$$

and obtain

$$B_2 = \frac{\mu_0 I}{4\pi} \int_0^\pi \frac{r d\theta}{r^2} = \frac{\mu_0 I}{4r} \quad (9.11.18)$$

The direction of $\vec{\mathbf{B}}_2$ is out of page, or $+\hat{\mathbf{k}}$.

(iii) The third segment of the wire runs from $(x, y) = (0, +r)$ to $(-\infty, +r)$. One may readily show that it gives the same contribution as the first one:

$$B_3 = B_1 = \frac{\mu_0 I}{4\pi r} \quad (9.11.19)$$

The direction of $\vec{\mathbf{B}}_3$ is again out of page, or $+\hat{\mathbf{k}}$.

The total magnitude of the magnetic field is

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 + \vec{\mathbf{B}}_3 = 2\vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 = \frac{\mu_0 I}{2\pi r} \hat{\mathbf{k}} + \frac{\mu_0 I}{4r} \hat{\mathbf{k}} = \frac{\mu_0 I}{4\pi r} (2 + \pi) \hat{\mathbf{k}} \quad (9.11.20)$$

Notice that the contribution from the two semi-infinite wires is equal to that due to an infinite wire:

$$\vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_3 = 2\vec{\mathbf{B}}_1 = \frac{\mu_0 I}{2\pi r} \hat{\mathbf{k}} \quad (9.11.21)$$

9.11.5 Two Infinitely Long Wires

Consider two infinitely long wires carrying currents are in the $-x$ -direction.

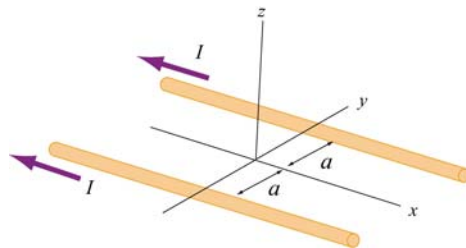


Figure 9.11.5 Two infinitely long wires

- (a) Plot the magnetic field pattern in the yz -plane.
- (b) Find the distance d along the z -axis where the magnetic field is a maximum.

Solutions:

- (a) The magnetic field lines are shown in Figure 9.11.6. Notice that the directions of both currents are into the page.

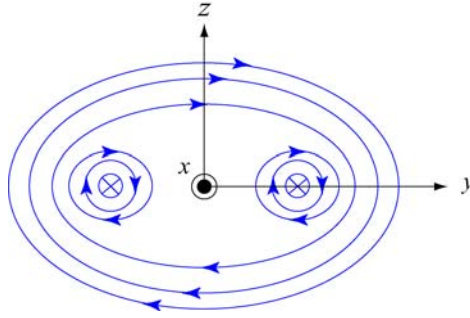


Figure 9.11.6 Magnetic field lines of two wires carrying current in the same direction.

- (b) The magnetic field at $(0, 0, z)$ due to wire 1 on the left is, using Ampere's law:

$$B_1 = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 I}{2\pi \sqrt{a^2 + z^2}} \quad (9.11.22)$$

Since the current is flowing in the $-x$ -direction, the magnetic field points in the direction of the cross product

$$(-\hat{\mathbf{i}}) \times \hat{\mathbf{r}}_1 = (-\hat{\mathbf{i}}) \times (\cos \theta \hat{\mathbf{j}} + \sin \theta \hat{\mathbf{k}}) = \sin \theta \hat{\mathbf{j}} - \cos \theta \hat{\mathbf{k}} \quad (9.11.23)$$

Thus, we have

$$\vec{\mathbf{B}}_1 = \frac{\mu_0 I}{2\pi \sqrt{a^2 + z^2}} (\sin \theta \hat{\mathbf{j}} - \cos \theta \hat{\mathbf{k}}) \quad (9.11.24)$$

For wire 2 on the right, the magnetic field strength is the same as the left one: $B_1 = B_2$. However, its direction is given by

$$(-\hat{\mathbf{i}}) \times \hat{\mathbf{r}}_2 = (-\hat{\mathbf{i}}) \times (-\cos \theta \hat{\mathbf{j}} + \sin \theta \hat{\mathbf{k}}) = \sin \theta \hat{\mathbf{j}} + \cos \theta \hat{\mathbf{k}} \quad (9.11.25)$$

Adding up the contributions from both wires, the z -components cancel (as required by symmetry), and we arrive at

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 = \frac{\mu_0 I \sin \theta}{\pi \sqrt{a^2 + z^2}} \hat{\mathbf{j}} = \frac{\mu_0 I z}{\pi (a^2 + z^2)} \hat{\mathbf{j}} \quad (9.11.26)$$

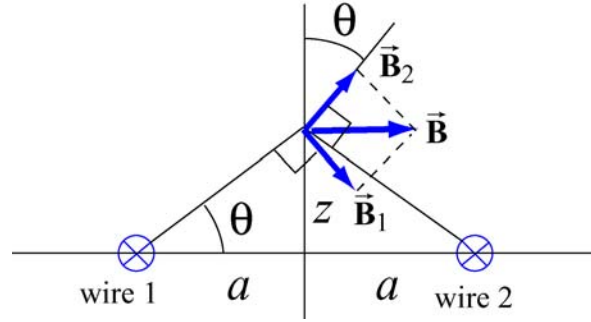


Figure 9.11.7 Superposition of magnetic fields due to two current sources

To locate the maximum of B , we set $dB/dz = 0$ and find

$$\frac{dB}{dz} = \frac{\mu_0 I}{\pi} \left(\frac{1}{a^2 + z^2} - \frac{2z^2}{(a^2 + z^2)^2} \right) = \frac{\mu_0 I}{\pi} \frac{a^2 - z^2}{(a^2 + z^2)^2} = 0 \quad (9.11.27)$$

which gives

$$z = a \quad (9.11.28)$$

Thus, at $z=a$, the magnetic field strength is a maximum, with a magnitude

$$B_{\max} = \frac{\mu_0 I}{2\pi a} \quad (9.11.29)$$

9.11.6 Non-Uniform Current Density

Consider an infinitely long, cylindrical conductor of radius R carrying a current I with a non-uniform current density

$$J = \alpha r \quad (9.11.30)$$

where α is a constant. Find the magnetic field everywhere.

(ii) Similarly, along the x -axis, we have $d\vec{s}_2 = dx'\hat{\mathbf{i}}$ and $\vec{r}_2 = (x-x')\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ which gives

$$d\vec{s}_2 \times \vec{r}_2 = y dx' \hat{\mathbf{k}} \quad (9.11.48)$$

Thus, we see that the magnetic field at P points in the $+z$ -direction. Using the above results and $r_1 = \sqrt{x^2 + (y-y')^2}$ and $r_2 = \sqrt{(x-x')^2 + y^2}$, we obtain

$$B_z = \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{x dy'}{[x^2 + (y-y')^2]^{3/2}} + \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{y dx'}{[y^2 + (x-x')^2]^{3/2}} \quad (9.11.49)$$

The integrals can be readily evaluated using

$$\int_0^\infty \frac{b ds}{[b^2 + (a-s)^2]^{3/2}} = \frac{1}{b} + \frac{a}{b\sqrt{a^2 + b^2}} \quad (9.11.50)$$

The final expression for the magnetic field is given by

$$\vec{B} = \frac{\mu_0 I}{4\pi} \left[\frac{1}{x} + \frac{y}{x\sqrt{x^2 + y^2}} + \frac{1}{y} + \frac{x}{y\sqrt{x^2 + y^2}} \right] \hat{\mathbf{k}} \quad (9.11.51)$$

We may show that the result is consistent with Eq. (9.1.5)

9.12 Conceptual Questions

1. Compare and contrast Biot-Savart law in magnetostatics with Coulomb's law in electrostatics.
2. If a current is passed through a spring, does the spring stretch or compress? Explain.
3. How is the path of the integration of $\oint \vec{B} \cdot d\vec{s}$ chosen when applying Ampere's law?
4. Two concentric, coplanar circular loops of different diameters carry steady currents in the same direction. Do the loops attract or repel each other? Explain.
5. Suppose three infinitely long parallel wires are arranged in such a way that when looking at the cross section, they are at the corners of an equilateral triangle. Can currents be arranged (combination of flowing in or out of the page) so that all three wires (a) attract, and (b) repel each other? Explain.

9.13 Additional Problems

9.13.1 Application of Ampere's Law

The simplest possible application of Ampere's law allows us to calculate the magnetic field in the vicinity of a single infinitely long wire. Adding more wires with differing currents will check your understanding of Ampere's law.

(a) Calculate with Ampere's law the magnetic field, $|\vec{B}| = B(r)$, as a function of distance r from the wire, in the vicinity of an infinitely long straight wire that carries current I . Show with a sketch the integration path you choose and state explicitly how you use symmetry. What is the field at a distance of 10 mm from the wire if the current is 10 A?

(b) Eight parallel wires cut the page perpendicularly at the points shown. A wire labeled with the integer k ($k = 1, 2, \dots, 8$) bears the current $2k$ times I_0 (i.e., $I_k = 2k I_0$). For those with $k = 1$ to 4, the current flows up out of the page; for the rest, the current flows down into the page. Evaluate $\oint \vec{B} \cdot d\vec{s}$ along the closed path (see figure) in the direction indicated by the arrowhead. (Watch your signs!)

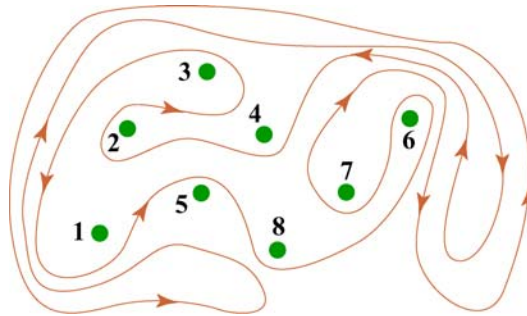


Figure 9.13.1 Amperian loop

(c) Can you use a single application of Ampere's Law to find the field at a point in the vicinity of the 8 wires? Why? How would you proceed to find the field at an arbitrary point P ?

9.13.2 Magnetic Field of a Current Distribution from Ampere's Law

Consider the cylindrical conductor with a hollow center and copper walls of thickness $b - a$ as shown in Figure 9.13.2. The radii of the inner and outer walls are a and b respectively, and the current I is uniformly spread over the cross section of the copper.

9.13.4 The Magnetic Field Through a Solenoid

A solenoid has 200 closely spaced turns so that, for most of its length, it may be considered to be an ideal solenoid. It has a length of 0.25 m, a diameter of 0.1 m, and carries a current of 0.30 A.

- (a) Sketch the solenoid, showing clearly the rotation direction of the windings, the current direction, and the magnetic field lines (inside and outside) with arrows to show their direction. What is the dominant direction of the magnetic field inside the solenoid?
- (b) Find the magnitude of the magnetic field inside the solenoid by constructing an Amperian loop and applying Ampere's law.
- (c) Does the magnetic field have a component in the direction of the wire in the loops making up the solenoid? If so, calculate its magnitude both inside and outside the solenoid, at radii 30 mm and 60 mm respectively, and show the directions on your sketch.

9.13.5 Rotating Disk

A circular disk of radius R with uniform charge density σ rotates with an angular speed ω . Show that the magnetic field at the center of the disk is

$$B = \frac{1}{2} \mu_0 \sigma \omega R$$

Hint: Consider a circular ring of radius r and thickness dr . Show that the current in this element is $dI = (\omega / 2\pi) dq = \omega \sigma r dr$.

9.13.6 Four Long Conducting Wires

Four infinitely long parallel wires carrying equal current I are arranged in such a way that when looking at the cross section, they are at the corners of a square, as shown in Figure 9.13.5. Currents in A and D point out of the page, and into the page at B and C . What is the magnetic field at the center of the square?

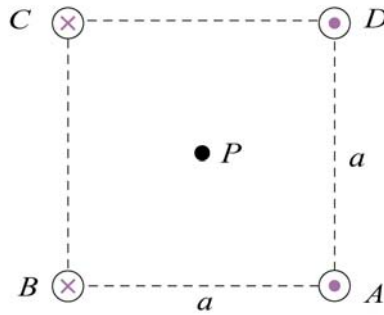


Figure 9.13.5 Four parallel conducting wires

9.13.7 Magnetic Force on a Current Loop

A rectangular loop of length l and width w carries a steady current I_1 . The loop is then placed near an infinitely long wire carrying a current I_2 , as shown in Figure 9.13.6. What is the magnetic force experienced by the loop due to the magnetic field of the wire?

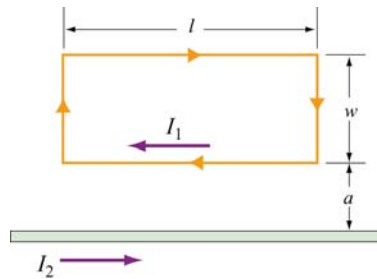


Figure 9.13.6 Magnetic force on a current loop.

9.13.8 Magnetic Moment of an Orbital Electron

We want to estimate the magnetic dipole moment associated with the motion of an electron as it orbits a proton. We use a “semi-classical” model to do this. Assume that the electron has speed v and orbits a proton (assumed to be very massive) located at the origin. The electron is moving in a right-handed sense with respect to the z -axis in a circle of radius $r = 0.53 \text{ \AA}$, as shown in Figure 9.13.7. Note that $1 \text{ \AA} = 10^{-10} \text{ m}$.

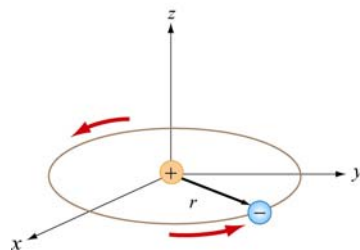


Figure 9.13.7