

## Summary of vector calculus in electrostatics (2)

- Laplacian:  $\nabla^2 \phi \equiv \nabla \cdot \nabla \phi$ 
  - In E&M:
    - Poisson Equation:  $\nabla^2 \phi = -4\pi\rho$
    - Laplace Equation:  $\nabla^2 \phi = 0$
    - Earnshaw's theorem: impossible to hold a charge in stable equilibrium with electrostatic fields (no local minima)

Comment:

This may look like a lot of math: it is!  
Time and exercise will help you to learn how to use it in E&M

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Purcell Chapter 2

## Conductors and Insulators

Conductor: a material with free electrons

- Excellent conductors: metals such as Au, Ag, Cu, Al,...
- OK conductors: ionic solutions such as NaCl in H<sub>2</sub>O



Insulator: a material without free electrons

- Organic materials: rubber, plastic,...
- Inorganic materials: quartz, glass,...

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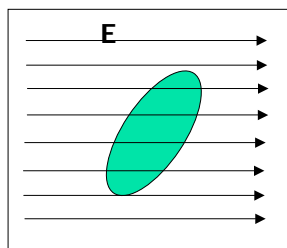
## Electric Fields in Conductors (1)

- A conductor is assumed to have an infinite supply of electric charges

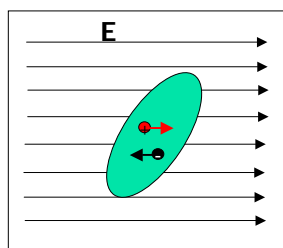
- Pretty good assumption...

- Inside a conductor,  $\mathbf{E}=0$

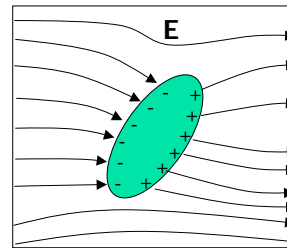
- Why? If  $\mathbf{E}$  is not 0  $\rightarrow$  charges will move from where the potential is higher to where the potential is lower; migration will stop only when  $\mathbf{E}=0$ .
  - How long does it take?  $10^{-17}$ - $10^{-16}$  s (typical resistivity of metals)



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## Electric Fields in Conductors (2)

- Electric potential inside a conductor is constant

- Given 2 points inside the conductor  $P_1$  and  $P_2$  the  $\Delta\phi$  would be:

$$\Delta\phi = \int_{P_1}^{P_2} \vec{E} \cdot d\vec{s} = 0 \quad \text{since } \mathbf{E}=0 \text{ inside the conductor.}$$

- Net charge can only reside on the surface

- If net charge inside the conductor  $\rightarrow$  Electric Field  $\neq 0$  (Gauss's law)

- External field lines are perpendicular to surface

- E// component would cause charge flow on the surface until  $\Delta\phi=0$

- Conductor's surface is an equipotential

- Because it's perpendicular to field lines

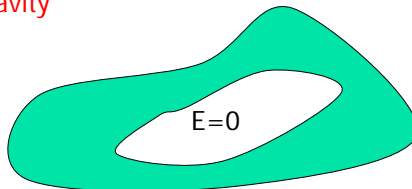
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## Corollary 1

In a hollow region inside conductor,  $\phi = \text{const}$  and  $E=0$  if there aren't any charges in the cavity



Why?

- Surface of conductor is equipotential
- If no charge inside the cavity  $\rightarrow$  Laplace holds  $\rightarrow \phi_{\text{cavity}}$  cannot have max or minima

$\rightarrow \phi$  must be constant  $\rightarrow E=0$

Consequence:

- Shielding of external electric fields: Faraday's cage

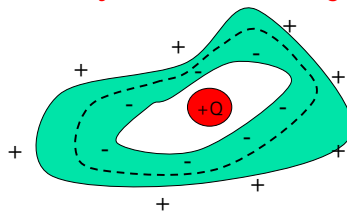
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## Corollary 2

A charge  $+Q$  in the cavity will induce a charge  $+Q$  on the outside of the conductor



Why?

- Apply Gauss's law to surface - - - inside the conductor

$$\oint \vec{E} \cdot d\vec{A} = 0 \text{ because } E=0 \text{ inside a conductor}$$

$$\oint \vec{E} \cdot d\vec{A} = 4\pi(Q + Q_{\text{inside}}) \text{ Gauss's law}$$

$$\Rightarrow Q_{\text{inside}} = -Q \Rightarrow Q_{\text{outside}} = -Q_{\text{inside}} = Q \text{ (conductor is overall neutral)}$$

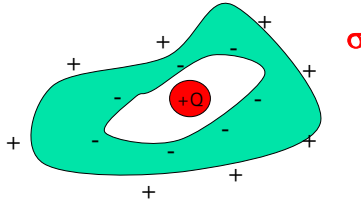
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## Corollary 3

The induced charge density on the surface of a conductor caused by a charge  $Q$  inside it is  $\sigma_{\text{induced}} = E_{\text{surface}}/4\pi$



Why?

- For surface charge layer, Gauss tells us that  $\Delta E = 4\pi\sigma$
- Since  $E_{\text{inside}} = 0 \rightarrow E_{\text{surface}} = 4\pi\sigma_{\text{induced}}$

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## Uniqueness theorem

Given the charge density  $\rho(x,y,z)$  in a region and the value of the electrostatic potential  $\phi(x,y,z)$  on the boundaries, there is only one function  $\phi(x,y,z)$  which describes the potential in that region.

Prove:

- Assume there are 2 solutions:  $\phi_1$  and  $\phi_2$ ; they will satisfy Poisson:

$$\nabla^2 \phi_1(\vec{r}) = 4\pi\rho(\vec{r})$$

$$\nabla^2 \phi_2(\vec{r}) = 4\pi\rho(\vec{r})$$

- Both  $\phi_1$  and  $\phi_2$  satisfy boundary conditions: on the boundary,  $\phi_1 = \phi_2 = \phi$
- Superposition: any combination of  $\phi_1$  and  $\phi_2$  will be solution, including  $\phi_3 = \phi_2 - \phi_1$ :  

$$\nabla^2 \phi_3(\vec{r}) = \nabla^2 \phi_2(\vec{r}) - \nabla^2 \phi_1(\vec{r}) = 4\pi\rho(\vec{r}) - 4\pi\rho(\vec{r}) = 0$$
- $\phi_3$  satisfies Laplace: no local maxima or minima inside the boundaries
- On the boundaries  $\phi_3 = 0 \rightarrow \phi_3 = 0$  everywhere inside region  

$$\rightarrow \phi_1 = \phi_2 \text{ everywhere inside region}$$

Why do I care?  
A solution is THE solution!

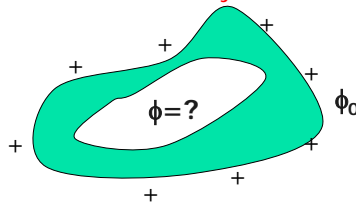
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## Uniqueness theorem: application 1

- A hollow conductor is charged until its external surface reaches a potential (relative to infinity)  $\phi = \phi_0$ .

What is the potential inside the cavity?



### Solution

$\phi = \phi_0$  everywhere inside the conductor's surface, including the cavity.

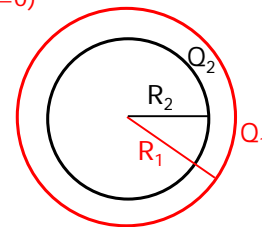
Why?  $\phi = \phi_0$  satisfies boundary conditions and Laplace equation

→ The uniqueness theorem tells me that is THE solution.

## Uniqueness theorem: application 2

- Two concentric thin conductive spherical shells of radii  $R_1$  and  $R_2$  carry charges  $Q_1$  and  $Q_2$  respectively.

- What is the potential of the outer sphere? ( $\phi_{\text{infinity}} = 0$ )
- What is the potential on the inner sphere?
- What at  $r = 0$ ?



### Solution

- Outer sphere:  $\phi_1 = (Q_1 + Q_2)/R_1$

- Inner sphere  $\phi_2 - \phi_1 = - \int_{R_2}^{R_1} \vec{E} \cdot d\vec{s} = - \int_{R_2}^{R_1} \frac{Q_2}{r^2} dr = \frac{Q_2}{R_1} - \frac{Q_2}{R_2}$

$$\Rightarrow \phi_2 = \frac{Q_2}{R_2} + \frac{Q_1}{R_1} \quad \text{Because of uniqueness: } \phi(r) = \phi_2 \forall r < R_2$$

## Next time...

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- More on Conductors in Electrostatics
- Capacitors
- NB: All these topics are included in Quiz 1  
scheduled for Tue October 5: just 2 weeks from now!!!
- Reminders:
  - Lab 1 is scheduled for Tomorrow 5-8 pm
  - Pset 2 is due THIS Fri Sep 24