

$$\begin{aligned}\vec{\mathbf{E}}(r, \theta) &= \frac{P}{4\pi\epsilon_0 r^3} \left[ 2\cos\theta (\sin\theta \hat{\mathbf{i}} + \cos\theta \hat{\mathbf{j}}) + \sin\theta \cos\theta \hat{\mathbf{i}} + (\cos^2\theta - 1)\hat{\mathbf{j}} \right] \\ &= \frac{P}{4\pi\epsilon_0 r^3} \left[ 2\cos\theta (\sin\theta \hat{\mathbf{i}} + \cos\theta \hat{\mathbf{j}}) + \sin\theta (\cos\theta \hat{\mathbf{i}} - \sin\theta \hat{\mathbf{j}}) \right]\end{aligned}$$

where the trigonometric identity  $(\cos^2\theta - 1) = -\sin^2\theta$  has been used. Since the unit vectors  $\hat{\mathbf{r}}$  and  $\hat{\boldsymbol{\theta}}$  in polar coordinates can be decomposed as

$$\begin{aligned}\hat{\mathbf{r}} &= \sin\theta \hat{\mathbf{i}} + \cos\theta \hat{\mathbf{j}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta \hat{\mathbf{i}} - \sin\theta \hat{\mathbf{j}},\end{aligned}$$

the electric field in polar coordinates is given by

$$\boxed{\vec{\mathbf{E}}(r, \theta) = \frac{P}{4\pi\epsilon_0 r^3} [2\cos\theta \hat{\mathbf{r}} + \sin\theta \hat{\boldsymbol{\theta}}]}$$

and the magnitude of  $\vec{\mathbf{E}}$  is

$$E = (E_r^2 + E_\theta^2)^{1/2} = \frac{P}{4\pi\epsilon_0 r^3} (3\cos^2\theta + 1)^{1/2}$$

### 2.13.5 Electric Field of an Arc

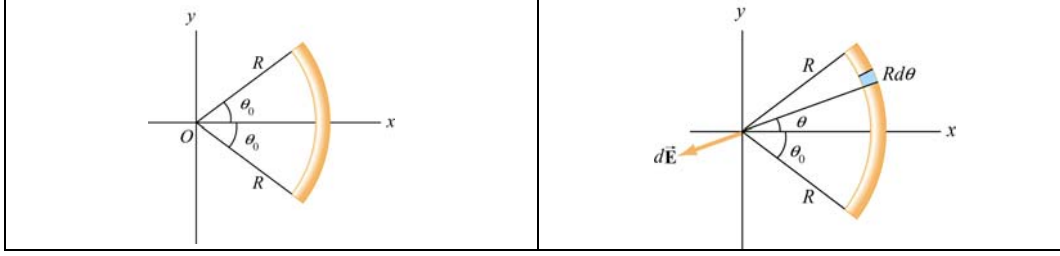
A thin rod with a uniform charge per unit length  $\lambda$  is bent into the shape of an arc of a circle of radius  $R$ . The arc subtends a total angle  $2\theta_0$ , symmetric about the  $x$ -axis, as shown in Figure 2.13.2. What is the electric field  $\vec{\mathbf{E}}$  at the origin  $O$ ?

**Solution:**

Consider a differential element of length  $d\ell = R d\theta$ , which makes an angle  $\theta$  with the  $x$ -axis, as shown in Figure 2.13.2(b). The amount of charge it carries is  $dq = \lambda d\ell = \lambda R d\theta$ .

The contribution to the electric field at  $O$  is

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{dq}{R^2} (-\cos\theta \hat{\mathbf{i}} - \sin\theta \hat{\mathbf{j}}) = \frac{1}{4\pi\epsilon_0} \frac{\lambda d\theta}{R} (-\cos\theta \hat{\mathbf{i}} - \sin\theta \hat{\mathbf{j}})$$



**Figure 2.13.2** (a) Geometry of charged source. (b) Charge element  $dq$

Integrating over the angle from  $-\theta_0$  to  $+\theta_0$ , we have

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\lambda}{R} \int_{-\theta_0}^{\theta_0} d\theta (-\cos\theta \hat{i} - \sin\theta \hat{j}) = \frac{1}{4\pi\epsilon_0} \frac{\lambda}{R} (-\sin\theta \hat{i} + \cos\theta \hat{j}) \Big|_{-\theta_0}^{\theta_0} = -\frac{1}{4\pi\epsilon_0} \frac{2\lambda \sin\theta_0}{R} \hat{i}$$

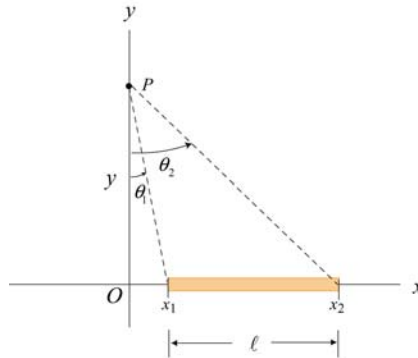
We see that the electric field only has the  $x$ -component, as required by a symmetry argument. If we take the limit  $\theta_0 \rightarrow \pi$ , the arc becomes a circular ring. Since  $\sin\pi = 0$ , the equation above implies that the electric field at the center of a non-conducting ring is zero. This is to be expected from symmetry arguments. On the other hand, for very small  $\theta_0$ ,  $\sin\theta_0 \approx \theta_0$  and we recover the point-charge limit:

$$\vec{E} \approx -\frac{1}{4\pi\epsilon_0} \frac{2\lambda\theta_0}{R} \hat{i} = -\frac{1}{4\pi\epsilon_0} \frac{2\lambda\theta_0 R}{R^2} \hat{i} = -\frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \hat{i}$$

where the total charge on the arc is  $Q = \lambda\ell = \lambda(2R\theta_0)$ .

### 2.13.6 Electric Field Off the Axis of a Finite Rod

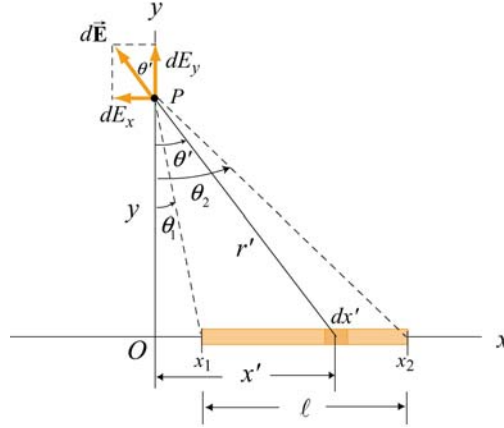
A non-conducting rod of length  $\ell$  with a uniform charge density  $\lambda$  and a total charge  $Q$  is lying along the  $x$ -axis, as illustrated in Figure 2.13.3. Compute the electric field at a point  $P$ , located at a distance  $y$  off the axis of the rod.



**Figure 2.13.3**

**Solution:**

The problem can be solved by following the procedure used in Example 2.3. Consider a length element  $dx'$  on the rod, as shown in Figure 2.13.4. The charge carried by the element is  $dq = \lambda dx'$ .



**Figure 2.13.4**

The electric field at  $P$  produced by this element is

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r'^2} \hat{r} = \frac{1}{4\pi\epsilon_0} \frac{\lambda dx'}{x'^2 + y^2} (-\sin \theta' \hat{i} + \cos \theta' \hat{j})$$

where the unit vector  $\hat{r}$  has been written in Cartesian coordinates:  $\hat{r} = -\sin \theta' \hat{i} + \cos \theta' \hat{j}$ . In the absence of symmetry, the field at  $P$  has both the  $x$ - and  $y$ -components. The  $x$ -component of the electric field is

$$dE_x = -\frac{1}{4\pi\epsilon_0} \frac{\lambda dx'}{x'^2 + y^2} \sin \theta' = -\frac{1}{4\pi\epsilon_0} \frac{\lambda dx'}{x'^2 + y^2} \frac{x'}{\sqrt{x'^2 + y^2}} = -\frac{1}{4\pi\epsilon_0} \frac{\lambda x' dx'}{(x'^2 + y^2)^{3/2}}$$

Integrating from  $x' = x_1$  to  $x' = x_2$ , we have

$$\begin{aligned} E_x &= -\frac{\lambda}{4\pi\epsilon_0} \int_{x_1}^{x_2} \frac{x' dx'}{(x'^2 + y^2)^{3/2}} = -\frac{\lambda}{4\pi\epsilon_0} \frac{1}{2} \int_{x_1^2 + y^2}^{x_2^2 + y^2} \frac{du}{u^{3/2}} = \frac{\lambda}{4\pi\epsilon_0} u^{-1/2} \Big|_{x_1^2 + y^2}^{x_2^2 + y^2} \\ &= \frac{\lambda}{4\pi\epsilon_0} \left[ \frac{1}{\sqrt{x_2^2 + y^2}} - \frac{1}{\sqrt{x_1^2 + y^2}} \right] = \frac{\lambda}{4\pi\epsilon_0 y} \left[ \frac{y}{\sqrt{x_2^2 + y^2}} - \frac{y}{\sqrt{x_1^2 + y^2}} \right] \\ &= \frac{\lambda}{4\pi\epsilon_0 y} (\cos \theta_2 - \cos \theta_1) \end{aligned}$$

Similarly, the  $y$ -component of the electric field due to the charge element is

$$dE_y = \frac{1}{4\pi\epsilon_0} \frac{\lambda dx'}{x'^2 + y^2} \cos \theta' = \frac{1}{4\pi\epsilon_0} \frac{\lambda dx'}{x'^2 + y^2} \frac{y}{\sqrt{x'^2 + y^2}} = \frac{1}{4\pi\epsilon_0} \frac{\lambda y dx'}{(x'^2 + y^2)^{3/2}}$$

Integrating over the entire length of the rod, we obtain

$$E_y = \frac{\lambda y}{4\pi\epsilon_0} \int_{x_1}^{x_2} \frac{dx'}{(x'^2 + y^2)^{3/2}} = \frac{\lambda y}{4\pi\epsilon_0} \frac{1}{y^2} \int_{\theta_1}^{\theta_2} \cos \theta' d\theta' = \frac{\lambda}{4\pi\epsilon_0 y} (\sin \theta_2 - \sin \theta_1)$$

where we have used the result obtained in Eq. (2.10.8) in completing the integration.

In the infinite length limit where  $x_1 \rightarrow -\infty$  and  $x_2 \rightarrow +\infty$ , with  $x_i = y \tan \theta_i$ , the corresponding angles are  $\theta_1 = -\pi/2$  and  $\theta_2 = +\pi/2$ . Substituting the values into the expressions above, we have

$$E_x = 0, \quad E_y = \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{y}$$

in complete agreement with the result shown in Eq. (2.10.11).

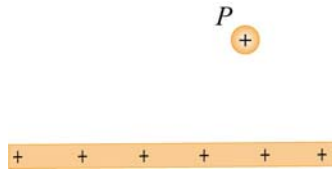
## 2.14 Conceptual Questions

1. Compare and contrast Newton's law of gravitation,  $F_g = Gm_1m_2/r^2$ , and Coulomb's law,  $F_e = kq_1q_2/r^2$ .
2. Can electric field lines cross each other? Explain.
3. Two opposite charges are placed on a line as shown in the figure below.



The charge on the right is three times the magnitude of the charge on the left. Besides infinity, where else can electric field possibly be zero?

4. A test charge is placed at the point  $P$  near a positively-charged insulating rod.



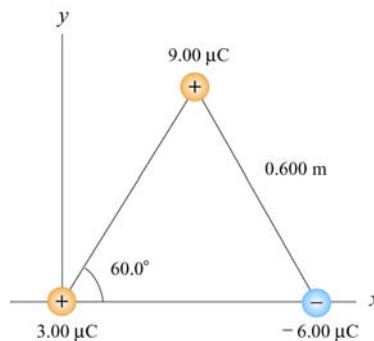
How would the magnitude and direction of the electric field change if the magnitude of the test charge were decreased and its sign changed with everything else remaining the same?

5. An electric dipole, consisting of two equal and opposite point charges at the ends of an insulating rod, is free to rotate about a pivot point in the center. The rod is then placed in a non-uniform electric field. Does it experience a force and/or a torque?

## 2.15 Additional Problems

### 2.15.1 Three Point Charges

Three point charges are placed at the corners of an equilateral triangle, as shown in Figure 2.15.1.

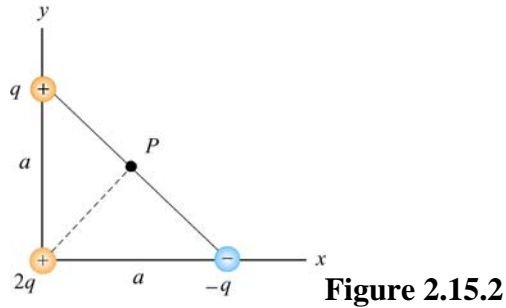


**Figure 2.15.1** Three point charges

Calculate the net electric force experienced by (a) the  $9.00 \mu\text{C}$  charge, and (b) the  $-6.00 \mu\text{C}$  charge.

### 2.15.2 Three Point Charges

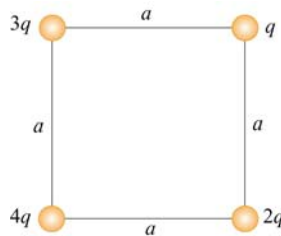
A right isosceles triangle of side  $a$  has charges  $q$ ,  $+2q$  and  $-q$  arranged on its vertices, as shown in Figure 2.15.2.



What is the electric field at point  $P$ , midway between the line connecting the  $+q$  and  $-q$  charges? Give the magnitude and direction of the electric field.

### 2.15.3 Four Point Charges

Four point charges are placed at the corners of a square of side  $a$ , as shown in Figure 2.15.3.

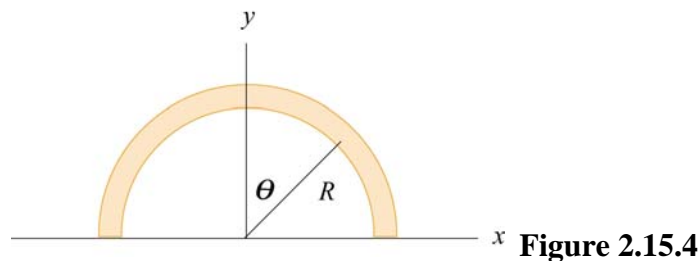


**Figure 2.15.3** Four point charges

- (a) What is the electric field at the location of charge  $q$  ?
- (b) What is the net force on  $2q$ ?

### 2.15.4 Semicircular Wire

A positively charged wire is bent into a semicircle of radius  $R$ , as shown in Figure 2.15.4.



The total charge on the semicircle is  $Q$ . However, the charge per unit length along the semicircle is non-uniform and given by  $\lambda = \lambda_0 \cos \theta$ .

(a) What is the relationship between  $\lambda_0$ ,  $R$  and  $Q$ ?

(b) If a charge  $q$  is placed at the origin, what is the total force on the charge?

### 2.15.5 Electric Dipole

An electric dipole lying in the  $xy$ -plane with a uniform electric field applied in the  $+x$ -direction is displaced by a small angle  $\theta$  from its equilibrium position, as shown in Figure 2.15.5.

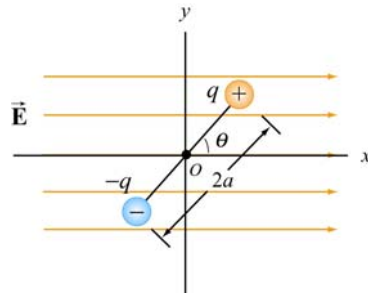


Figure 2.15.5

The charges are separated by a distance  $2a$ , and the moment of inertia of the dipole is  $I$ . If the dipole is released from this position, show that its angular orientation exhibits simple harmonic motion. What is the frequency of oscillation?

### 2.15.6 Charged Cylindrical Shell and Cylinder

(a) A uniformly charged circular *cylindrical shell* of radius  $R$  and height  $h$  has a total charge  $Q$ . What is the electric field at a point  $P$  a distance  $z$  from the bottom side of the cylinder as shown in Figure 2.15.6? (*Hint*: Treat the cylinder as a set of ring charges.)

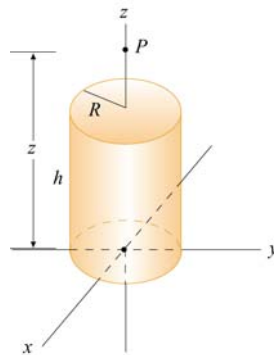
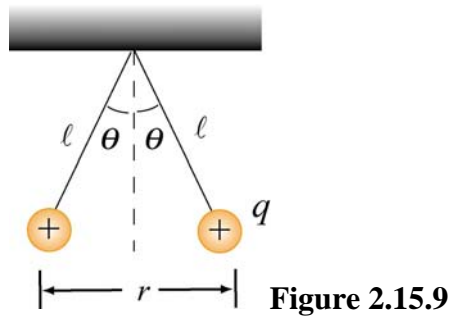


Figure 2.15.6 A uniformly charged cylinder

(b) If the configuration is instead a *solid cylinder* of radius  $R$ , height  $h$  and has a uniform volume charge density. What is the electric field at  $P$ ? (*Hint*: Treat the solid cylinder as a set of disk charges.)

### 2.15.7 Two Conducting Balls

Two tiny conducting balls of identical mass  $m$  and identical charge  $q$  hang from non-conducting threads of length  $\ell$ . Each ball forms an angle  $\theta$  with the vertical axis, as shown in Figure 2.15.9. Assume that  $\theta$  is so small that  $\tan\theta \approx \sin\theta$ .



(a) Show that, at equilibrium, the separation between the balls is

$$r = \left( \frac{q^2 \ell}{2\pi\epsilon_0 mg} \right)^{1/3}$$

(b) If  $\ell = 1.2 \times 10^2$  cm,  $m = 1.0 \times 10^1$  g, and  $x = 5.0$  cm, what is  $q$ ?

### 2.15.8 Torque on an Electric Dipole

An electric dipole consists of two charges  $q_1 = +2e$  and  $q_2 = -2e$  ( $e = 1.6 \times 10^{-19}$  C), separated by a distance  $d = 10^{-9}$  m. The electric charges are placed along the  $y$ -axis as shown in Figure 2.15.10.

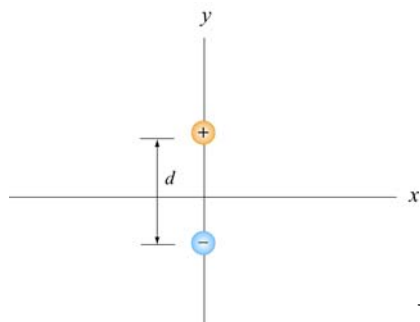


Figure 2.15.10