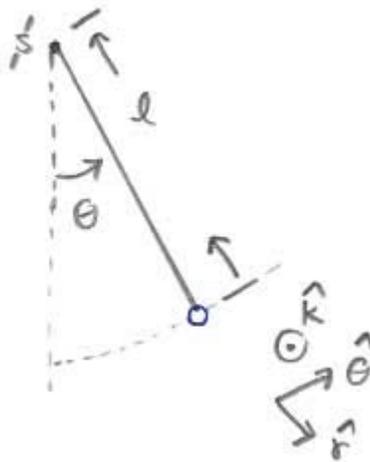


Problem 31: Simple Pendulum

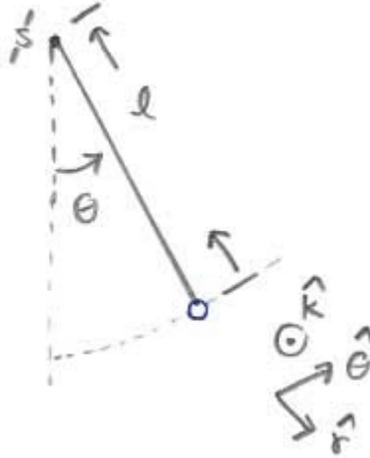
A pendulum consists of an object hanging from the end of a string. The object is pulled to one side and allowed to oscillate. If the object has negligible size and the string is massless, then the pendulum is called a simple pendulum.



- Find the equation of motion for the object.
- Show that the object undergoes simple harmonic motion when the initial angle is small and you can approximate $\sin \theta \cong \theta$.
- For small angles, what is the period of oscillation?
- Is the angular velocity the same as the angular frequency for the object?

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The torque about the pivot point is given by

$$\vec{\tau}_S = \vec{r}_{s,m} \times m\vec{g} = l\hat{r} \times mg(-\sin \theta \hat{\theta} + \cos \theta \hat{r}) = -lmg \sin \theta \hat{k}.$$

The angular acceleration is a vector pointing along the z-axis,

$$\vec{\alpha} = \frac{d^2\theta}{dt^2} \hat{k}.$$

Remark: The z-component of the angular acceleration vector is $\frac{d^2\theta}{dt^2}$ and can be positive or negative.

The moment of inertial of a point mass about the pivot point is

$$I_S = ml^2$$

The rotational dynamical equation is

$$\vec{\tau}_S = I_S \vec{\alpha}.$$

Therefore

$$-lmg \sin \theta \hat{\mathbf{k}} = ml^2 \frac{d^2 \theta}{dt^2} \hat{\mathbf{k}}.$$

Thus we have the equation of motion for the simple pendulum,

$$-lmg \sin \theta = ml^2 \frac{d^2 \theta}{dt^2}.$$

When the angle of oscillation is small, then we can use the small angle approximation

$$\sin \theta \cong \theta.$$

Then the pendulum equation becomes

$$\frac{d^2 \theta}{dt^2} \cong -\frac{g}{l} \theta.$$

This equation is similar to the mass-spring equation,

$$\frac{d^2 x}{dt^2} = -\frac{k}{m} x$$

that described the oscillation of a mass about the equilibrium point of a spring. Recall that the angular frequency of oscillation was given by

$$\omega_{spring} = \sqrt{\frac{k}{m}}.$$

So by comparison, the angular frequency of oscillation for the pendulum is approximately

$$\omega_{pendulum} \cong \sqrt{\frac{g}{l}},$$

with period

$$T = \frac{2\pi}{\omega_p} \cong 2\pi \sqrt{\frac{l}{g}}.$$

By analogy with the spring, if the pendulum has been displaced by a small angle θ_0 and released from rest, the angle that the pendulum makes with the vertical as a function of time is given by

$$\theta(t) = \theta_0 \cos((\sqrt{g/l})t).$$

The angular velocity of the pendulum is then given by the derivative

$$d\theta/dt = -(\sqrt{g/l})\theta_0 \sin((\sqrt{g/l})t).$$

This varies sinusoidally with time and is not equal to the angular frequency which is a constant associated with the periodic nature of the motion.