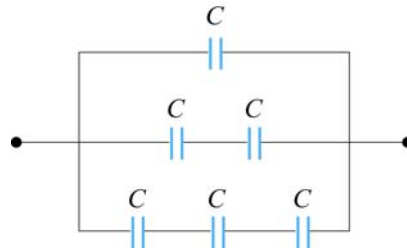


(4) Calculate $C$ using $C = Q /  \Delta V $	$C = \frac{\epsilon_0 A}{d}$	$C = \frac{2\pi\epsilon_0 l}{\ln(b/a)}$	$C = 4\pi\epsilon_0 \left( \frac{ab}{b-a} \right)$
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## 5.10 Solved Problems

### 5.10.1 Equivalent Capacitance

Consider the configuration shown in Figure 5.10.1. Find the equivalent capacitance, assuming that all the capacitors have the same capacitance  $C$ .



**Figure 5.10.1** Combination of Capacitors

#### Solution:

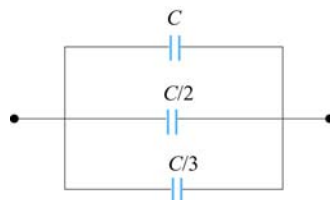
For capacitors that are connected in series, the equivalent capacitance is

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots = \sum_i \frac{1}{C_i} \quad (\text{series})$$

On the other hand, for capacitors that are connected in parallel, the equivalent capacitance is

$$C_{\text{eq}} = C_1 + C_2 + \cdots = \sum_i C_i \quad (\text{parallel})$$

Using the above formula for series connection, the equivalent configuration is shown in Figure 5.10.2.



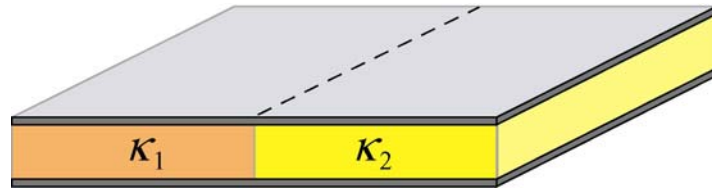
**Figure 5.10.2**

Now we have three capacitors connected in parallel. The equivalent capacitance is given by

$$C_{\text{eq}} = C \left( 1 + \frac{1}{2} + \frac{1}{3} \right) = \frac{11}{6} C$$

### 5.10.2 Capacitor Filled with Two Different Dielectrics

Two dielectrics with dielectric constants  $\kappa_1$  and  $\kappa_2$  each fill half the space between the plates of a parallel-plate capacitor as shown in Figure 5.10.3.



**Figure 5.10.3** Capacitor filled with two different dielectrics.

Each plate has an area  $A$  and the plates are separated by a distance  $d$ . Compute the capacitance of the system.

**Solution:**

Since the potential difference on each half of the capacitor is the same, we may treat the system as being composed of two capacitors connected in parallel. Thus, the capacitance of the system is

$$C = C_1 + C_2$$

With

$$C_i = \frac{\kappa_i \epsilon_0 (A/2)}{d}, \quad i = 1, 2$$

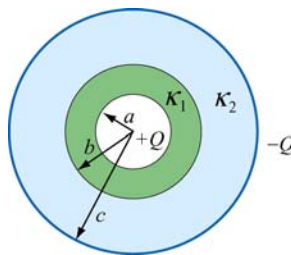
we obtain

$$C = \frac{\kappa_1 \epsilon_0 (A/2)}{d} + \frac{\kappa_2 \epsilon_0 (A/2)}{d} = \frac{\epsilon_0 A}{2d} (\kappa_1 + \kappa_2)$$

### 5.10.3 Capacitor with Dielectrics

Consider a conducting spherical shell with an inner radius  $a$  and outer radius  $c$ . Let the space between two surfaces be filled with two different dielectric materials so that the

dielectric constant is  $\kappa_1$  between  $a$  and  $b$ , and  $\kappa_2$  between  $b$  and  $c$ , as shown in Figure 5.10.4. Determine the capacitance of this system.



**Figure 5.10.4** Spherical capacitor filled with dielectrics.

**Solution:**

The system can be treated as two capacitors connected in series, since the total potential difference across the capacitors is the sum of potential differences across individual capacitors. The equivalent capacitance for a spherical capacitor of inner radius  $r_1$  and outer radius  $r_2$  filled with dielectric with dielectric constant  $\kappa_e$  is given by

$$C = 4\pi\epsilon_0\kappa_e \left( \frac{r_1 r_2}{r_2 - r_1} \right)$$

Thus, the equivalent capacitance of this system is

$$\frac{1}{C} = \frac{1}{\frac{4\pi\epsilon_0\kappa_1 ab}{(b-a)}} + \frac{1}{\frac{4\pi\epsilon_0\kappa_2 bc}{(c-b)}} = \frac{\kappa_2 c(b-a) + \kappa_1 a(c-b)}{4\pi\epsilon_0\kappa_1\kappa_2 abc}$$

or

$$C = \frac{4\pi\epsilon_0\kappa_1\kappa_2 abc}{\kappa_2 c(b-a) + \kappa_1 a(c-b)}$$

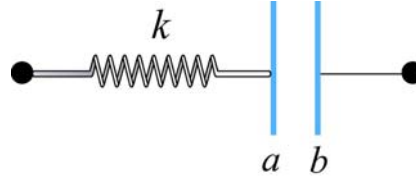
It is instructive to check the limit where  $\kappa_1, \kappa_2 \rightarrow 1$ . In this case, the above expression reduces to

$$C = \frac{4\pi\epsilon_0 abc}{c(b-a) + a(c-b)} = \frac{4\pi\epsilon_0 abc}{b(c-a)} = \frac{4\pi\epsilon_0 ac}{(c-a)}$$

which agrees with Eq. (5.2.11) for a spherical capacitor of inner radius  $a$  and outer radius  $c$ .

#### 5.10.4 Capacitor Connected to a Spring

Consider an air-filled parallel-plate capacitor with one plate connected to a spring having a force constant  $k$ , and another plate held fixed. The system rests on a table top as shown in Figure 5.10.5.



**Figure 5.10.5** Capacitor connected to a spring.

If the charges placed on plates  $a$  and  $b$  are  $+Q$  and  $-Q$ , respectively, how much does the spring expand?

**Solution:**

The spring force  $\vec{F}_s$  acting on plate  $a$  is given by

$$\vec{F}_s = -kx \hat{i}$$

Similarly, the electrostatic force  $\vec{F}_e$  due to the electric field created by plate  $b$  is

$$\vec{F}_e = QE \hat{i} = Q \left( \frac{\sigma}{2\epsilon_0} \right) \hat{i} = \frac{Q^2}{2A\epsilon_0} \hat{i}$$

where  $A$  is the area of the plate. Notice that charges on plate  $a$  cannot exert a force on itself, as required by Newton's third law. Thus, only the electric field due to plate  $b$  is considered. At equilibrium the two forces cancel and we have

$$kx = Q \left( \frac{Q}{2A\epsilon_0} \right)$$

which gives

$$x = \frac{Q^2}{2kA\epsilon_0}$$

#### 5.11 Conceptual Questions

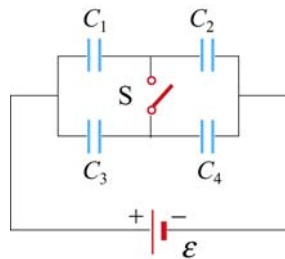
1. The charges on the plates of a parallel-plate capacitor are of opposite sign, and they attract each other. To increase the plate separation, is the external work done positive or negative? What happens to the external work done in this process?

2. How does the stored energy change if the potential difference across a capacitor is tripled?
3. Does the presence of a dielectric increase or decrease the maximum operating voltage of a capacitor? Explain.
4. If a dielectric-filled capacitor is cooled down, what happens to its capacitance?

## 5.12 Additional Problems

### 5.12.1 Capacitors in Series and in Parallel

A 12-Volt battery charges the four capacitors shown in Figure 5.12.1.



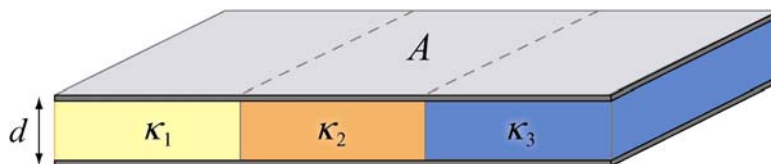
**Figure 5.12.1**

Let  $C_1 = 1 \mu\text{F}$ ,  $C_2 = 2 \mu\text{F}$ ,  $C_3 = 3 \mu\text{F}$ , and  $C_4 = 4 \mu\text{F}$ .

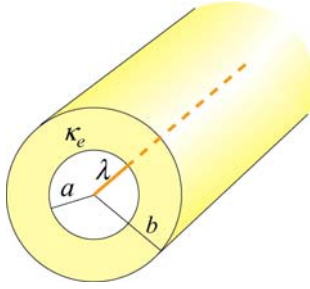
- (a) What is the equivalent capacitance of the group  $C_1$  and  $C_2$  if switch  $S$  is open (as shown)?
- (b) What is the charge on *each* of the four capacitors if switch  $S$  is open?
- (c) What is the charge on each of the four capacitors if switch  $S$  is closed?

### 5.12.2 Capacitors and Dielectrics

- (a) A parallel-plate capacitor of area  $A$  and spacing  $d$  is filled with three dielectrics as shown in Figure 5.12.2. Each occupies  $1/3$  of the volume. What is the capacitance of this system? [*Hint*: Consider an equivalent system to be three parallel capacitors, and justify this assumption.] Show that you obtain the proper limits as the dielectric constants approach unity,  $\kappa_i \rightarrow 1$ .]



**Figure 5.12.2**



**Figure 5.12.5**

The material has a dielectric constant  $\kappa_e = 10$ . At the center of the shell there is a line charge running parallel to the axis of the cylindrical shell, with free charge per unit length  $\lambda$ .

- (a) Find the electric field for:  $r < a$ ,  $a < r < b$  and  $r > b$ .
- (b) What is the induced surface charge per unit length on the inner surface of the spherical shell? [Ans:  $-9\lambda/10$ .]
- (c) What is the induced surface charge per unit length on the outer surface of the spherical shell? [Ans:  $+9\lambda/10$ .]

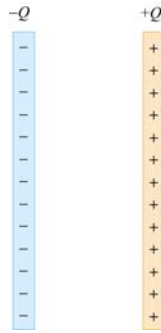
### 5.12.5 A Capacitor with a Dielectric

A parallel plate capacitor has a capacitance of 112 pF, a plate area of 96.5 cm<sup>2</sup>, and a mica dielectric ( $\kappa_e = 5.40$ ). At a 55 V potential difference, calculate

- (a) the electric field strength in the mica; [Ans: 13.4 kV/m.]
- (b) the magnitude of the free charge on the plates; [Ans: 6.16 nC.]
- (c) the magnitude of the induced surface charge; [Ans: 5.02 nC.]
- (d) the magnitude of the polarization  $\vec{P}$  [Ans: 520 nC/m<sup>2</sup>.]

### 5.12.6 Force on the Plates of a Capacitor

The plates of a parallel-plate capacitor have area  $A$  and carry total charge  $\pm Q$  (see Figure 5.12.6). We would like to show that these plates *attract* each other with a force given by  $F = Q^2/(2\epsilon_0 A)$ .



**Figure 5.12.6**

- (a) Calculate the total force on the left plate due to the electric field of the right plate, using Coulomb's Law. Ignore fringing fields.
- (b) If you pull the plates apart, against their attraction, you are doing work and *that work goes directly into creating additional electrostatic energy*. Calculate the force necessary to increase the plate separation from  $x$  to  $x+dx$  by equating the work you do,  $\vec{F} \cdot d\vec{x}$ , to the increase in electrostatic energy, assuming that the electric energy density is  $\epsilon_0 E^2/2$ , and that the charge  $Q$  remains constant.
- (c) Using this expression for the force, show that the force per unit area (the *electrostatic stress*) acting on either capacitor plate is given by  $\epsilon_0 E^2/2$ . This result is true for a conductor of any shape with an electric field  $\vec{E}$  at its surface.
- (d) Atmospheric pressure is 14.7 lb/in<sup>2</sup>, or 101,341 N/m<sup>2</sup>. How large would  $E$  have to be to produce this force per unit area? [Ans: 151 MV/m. Note that Van de Graff accelerators can reach fields of 100 MV/m maximum before breakdown, so that electrostatic stresses are on the same order as atmospheric pressures in this extreme situation, but not much greater].

### 5.12.7 Energy Density in a Capacitor with a Dielectric

Consider the case in which a dielectric material with dielectric constant  $\kappa_e$  completely fills the space between the plates of a parallel-plate capacitor. Show that the energy density of the field between the plates is  $u_E = \vec{E} \cdot \vec{D}/2$  by the following procedure:

- (a) Write the expression  $u_E = \vec{E} \cdot \vec{D}/2$  as a function of  $\mathbf{E}$  and  $\kappa_e$  (i.e. eliminate  $\vec{D}$ ).
- (b) Given the electric field and potential of such a capacitor with free charge  $q$  on it (problem 4-1a above), calculate the work done to charge up the capacitor from  $q=0$  to  $q=Q$ , the final charge.
- (c) Find the energy density  $u_E$ .