PKT ABSOLUTE

Rigid Body Kinematics

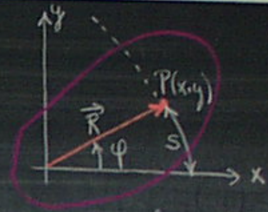
- Real Objects → mass dist. size, shape
- Motion: Translation of CM
  Rotation about axis.
- Inertial Frame

Point P moves in a circle of radius R about a fixed axis.

$S = R\varphi$ $\qquad \varphi(\text{rad}) = \frac{\pi}{180}\varphi(\text{deg})$

$\varphi = +\ \text{ccw}$, $-\ \text{cw}$ $\qquad \varphi = 0$ x-axis, $\varphi = \pi/2$ y-axis, $\varphi = 2\pi$ x-axis again

$\varphi \neq$ vector!!



In general $\varphi = \varphi(t)$

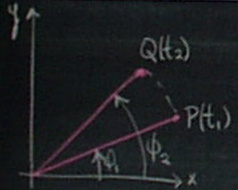
Angular Velocity: $\omega(t)$ ①

$$\vec{\omega} = \frac{\varphi_2 - \varphi_1}{t_2 - t_1} = \frac{\Delta\varphi}{\Delta t}\hat{k} \quad (s^{-1})$$

$\vec{\omega} \rightarrow$ along axis of rotation (RHR)

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} = \frac{d\varphi}{dt}\hat{k} \quad \text{rad/s}$$

$\Rightarrow$ Inst. Angular Velocity



Suppose $\omega(t) = \omega_0 =$ constant.

1 revolution $= 2\pi$ radians

Period: $T = \frac{2\pi}{\omega_0}$ (s)

Frequency: $\nu = \frac{1}{T} = \frac{\omega_0}{2\pi}$ (Hz)

$\nu = 1 \Rightarrow 1$ rev/s

$\nu = 10 \Rightarrow 10$ rev/s

etc.

Angular Acceleration: $\alpha(t)$

$$\vec{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1}\hat{k} = \frac{\Delta\omega}{\Delta t}\hat{k} \quad (s^{-2})$$

$$\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}\hat{k} = \frac{d\vec{\omega}}{dt} = \frac{d}{dt}\left(\frac{d\varphi}{dt}\right)\hat{k}$$

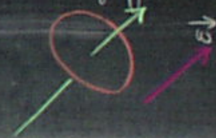
$\therefore \vec{\alpha}(t) = \frac{d^2\varphi}{dt^2}\hat{k}$ Inst. ang. accel.

Note: Every point: same $\omega(t)$!!!
same $\alpha(t)$

Fixed Rotation Axis: ②

direction $\vec{\alpha} \equiv$ dir $\vec{\omega}$

Changing Axis: No!!!

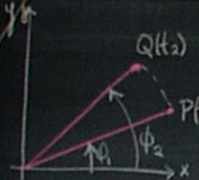


$\vec{\omega}$

$\frac{d\vec{\omega}}{dt} > 0$ $\vec{\alpha}$ same as $\vec{\omega}$

$\frac{d\vec{\omega}}{dt} < 0$ $\vec{\alpha}$ opposite $\vec{\omega}$

Lecture 21 Blackboard #1



Suppose $\omega(t) = \omega_0 = \text{constant}$.

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Note: Every point: same $\omega(t)$!!!
Same $\alpha(t)$!!!

Fixed Rotation Axis: ②

direction $\vec{\alpha} \equiv$ dir $\vec{\omega}$

Changing Axes: No!!!

$\dfrac{d\vec{\omega}}{dt} > 0$ $\vec{\alpha}$ same as $\vec{\omega}$

$\dfrac{d\vec{\omega}}{dt} < 0$ $\vec{\alpha}$ opposite to $\vec{\omega}$

Rotational Motion: $\vec{\alpha} \equiv$ constant

- Fixed Axes.
  Ignore vector notation ($\pm$ direction)
- Also true for axes in linear translation (rolling)

$$\frac{d\omega}{dt} = \alpha \Rightarrow \int_{\omega_0}^{\omega} d\omega' = \int_0^t \alpha\, dt'$$

$$\boxed{\omega = \omega_0 + \alpha t} \quad ①$$

$$\frac{d\varphi}{dt} = \omega = \omega_0 + \alpha t$$

$$\int_{\varphi_0}^{\varphi} d\varphi' = \int_0^t \omega_0\, dt' + \alpha\int_0^t t\, dt'$$

$$\boxed{\varphi = \varphi_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2} \quad ②$$

From ① $(\omega - \omega_0)/\alpha = t$

$$\rightarrow ② \quad \boxed{\omega^2 = \omega_0^2 + 2\alpha(\varphi - \varphi_0)} \quad ③$$

Angular $\longleftrightarrow$ linear Velocity and Acceleration ③

Every particle moves in a circle about axis of rot.

$s = R\varphi$

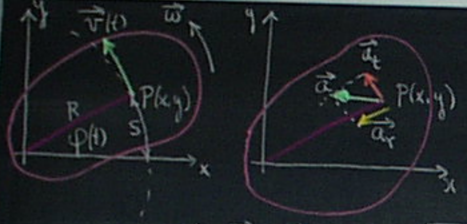
$v = \dfrac{ds}{dt} = R\dfrac{d\varphi}{dt} = R\omega$ [Prop. to distance from axis]

$a_t = a_{\parallel} = \dfrac{dv}{dt} = R\dfrac{d\omega}{dt} = R\alpha$ [Due to change in mag. of $\vec{v}$]

$a_r = a_{\perp} = \dfrac{v^2}{R} = \dfrac{R^2\omega^2}{R} = R\omega^2$ [Change in dir. of $\vec{v}$]

a_t, a_r: Prop to R !! If $\alpha = 0$, $a_t = 0$; $a_r \neq 0$

Lecture 21 Blackboard #2



$\vec a = \vec a_r + \vec a_t$ $\quad |a| = R\sqrt{\alpha^2 + \omega^4}$

Rotational Kinetic Energy

$K = \frac{1}{2}\sum_i m_i v_i^2$

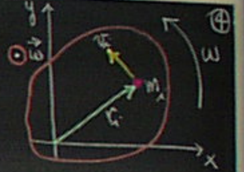
$v_i = r_i \omega$

$K = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$

$K = \frac{1}{2} I \omega^2$

$\boxed{I = \sum_i m_i r_i^2}$

- Moment-of-Inertia
- Depends on axis of rotation.
- Particles at large r_i contribute most!



$a, M \longleftrightarrow$ Resis. to Linear Motion

$\alpha, I \longleftrightarrow$ Resis. to Rotational Motion

} Inertial Quantities

Example: 4 Rotating Masses

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$\omega = \omega_0$: ang velocity constant.

$\alpha = 0$

a) Rotate about y-axis:

$I_y = \sum m_i r_i^2 = Ma^2 + Ma^2 + 0 + 0 = 2Ma^2$

$K_y = \frac{1}{2} I_y \omega^2 = \frac{1}{2}(2Ma^2)\omega_0^2$

b) Rotate about z-axis:

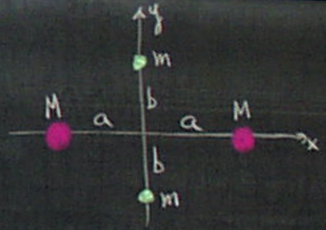
$I_z = \sum m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2$

$= 2Ma^2 + 2mb^2$

$K_z = \frac{1}{2} I_z \omega_0^2 = \frac{1}{2} 2(Ma^2 + mb^2)\omega_0^2$

$I_z > I_y$

$K_z > K_y$



Lecture 21 Blackboard #3

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Example: 4 Rotating Masses

$\omega = \omega_0$: ang. velocity constant.
$\alpha = 0$

a) Rotate about y-axis:

$$I_y = \sum m_i r_{\perp}^2 = Ma^2 + Ma^2 + 0 + 0 = 2Ma^2$$

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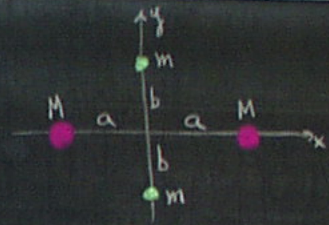
b) Rotate about z-axis:

$$I_z = \sum m_i r_{\perp}^2 = Ma^2 + Ma^2 + mb^2 + mb^2$$
$$= 2Ma^2 + 2mb^2$$

$$K_z = \tfrac{1}{2} I_z \omega_0^2 = \tfrac{1}{2} 2(Ma^2 + mb^2)\omega_0^2$$

$I_z > I_y$

$K_z > K_y$



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Rigid Bodies: Moments-of-Inertia:

$K = \tfrac{1}{2} I \omega^2$ KE for rotation about fixed axis.

Δm = element of mass.

$I = \sum r_{\perp}^2 \Delta m$ about rot. axis.

$$I = \lim_{\Delta m \to 0} \sum r_{\perp}^2 \Delta m = \int_V r^2 \, dm.$$

Let $\rho(x,y,z)$ = local density (mass/volume)

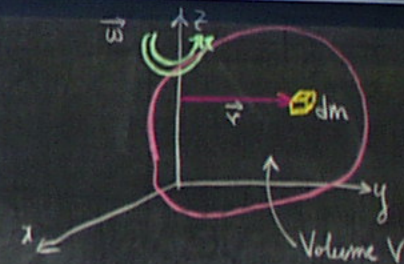
$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

$\therefore dm = \rho \, dV$

$$\boxed{I = \int_V \rho r^2 \, dV}$$

$I \Longrightarrow$ 2nd moment of mass dist.

If $\rho(x,y,z)$ = constant $\Longrightarrow$ $\boxed{I = \frac{M}{V}\int_V r^2 \, dV}$ where $\rho \equiv \frac{M}{V}$



Parallel-Axis Theorem

The moment-of-inertia depends on the location of the axis of rotation.

KE for a body about a fixed axis is

$$K = \frac{1}{2} I_z \omega^2$$

We had before that the KE is the sum of the translational energy of motion of the cm and the internal energy of motion relative to the cm.

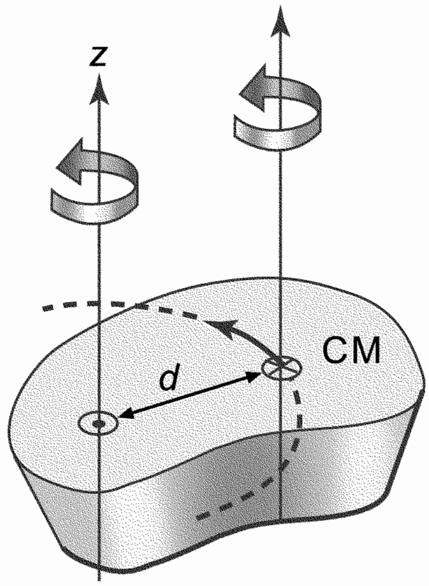
$$K = \frac{1}{2} M v_{cm}^2 + K_{Int}.$$

Consider rotation about z-axis not through cm.

cm moves in a circle of radius d around this axis.

$$\therefore v_{cm} = d\omega$$

$$\therefore \frac{1}{2} M v_{cm}^2 = \frac{1}{2} M d^2 \omega^2$$



Two alternative parallel axes of rotation of a rigid body. The z axis is fixed. The center-of-mass axis moves along a circle of radius *d* around the z axis. The body is in rotational motion relative to each of these axes.

The rotation of the body with angular velocity ω about a fixed z-axis is also a rotation about a parallel axis through the cm with the same angular velocity ω. One turn around z corresponds to one turn around axis through cm.

The KE associated with rotational motion around axis through cm is

$$K_{Int} = \frac{1}{2} I_{cm} \omega^2$$

$$\therefore K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M d^2 \omega^2$$

$$\frac{1}{2} I_z \omega^2 = \frac{1}{2}\left[I_{cm} + M d^2 \right] \omega^2$$

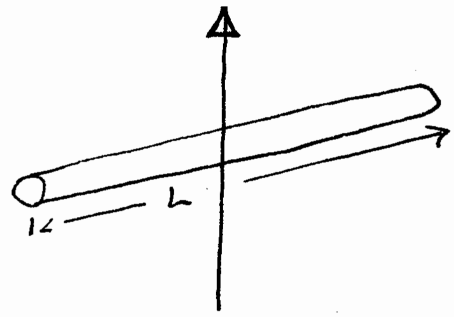
Comparing, we must have

$$I_z = I_{cm} + M d^2$$

[Parallel Axis Theorem]

Example: Rod

- Thin rod axis through its midpoint



$$I_{cm} = \frac{1}{12} ML^2$$

- what is I about an axis through its end ?

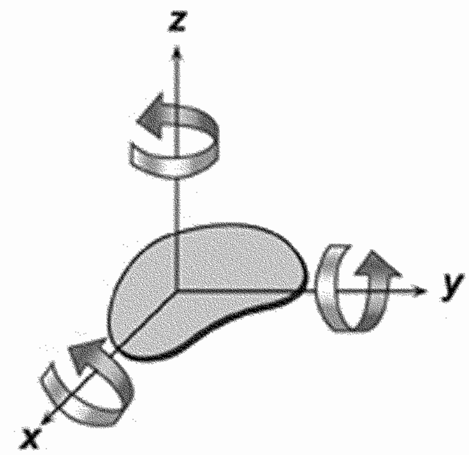
$$d = L/2$$

$$I = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2$$

$$I = \frac{ML^2}{3}$$

Perpindicular - Axis Theorem.

Relates moments-of-inertia of a thin flat plate about three mutually perpindicular axes.



A thin, flat plate. The plate may rotate about either the the x axis, the y axis, or the z axis.

Consider a thin plate which can rotate about any of three $\perp$ axes

$$\left.\begin{matrix} I_x \\ I_y \\ I_z \end{matrix}\right\} \text{corresponding moments-of-inertia}$$

Let plate be in xy-plane. The distance from z-axis to reference point P is

$$R = \sqrt{x^2 + y^2}$$

$$I_z = \int \rho R^2 dV = \int \rho\,(x^2+y^2)\,dV$$

$$I_x = \int \rho y^2 dV$$

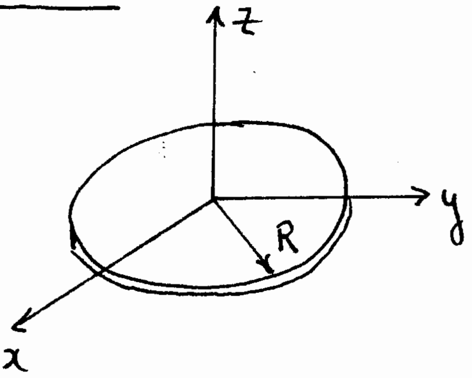
$$I_y = \int \rho x^2 dV$$

$$\therefore \boxed{I_z = I_x + I_y}$$

[Perpindicular-Axis Thm]

Example

i) Disk



• Disk in xy-plane

$$I_z = \frac{1}{2} M R^2$$

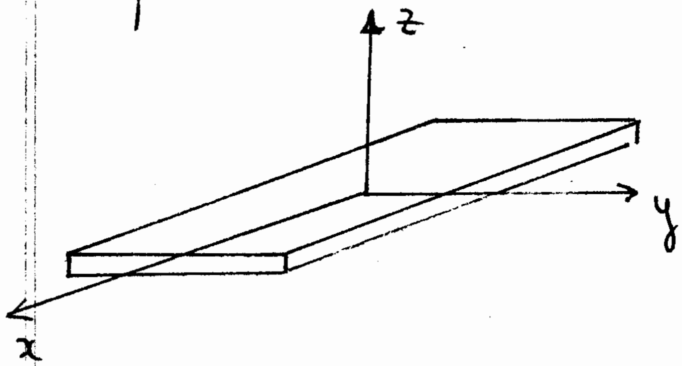
By symmetry

$$I_x = I_y$$

$$\therefore \quad I_z = 2 I_x$$

$$I_x = I_y = \frac{I_z}{2} = \frac{MR^2}{4}$$

ii) Square Plate.



Side = a

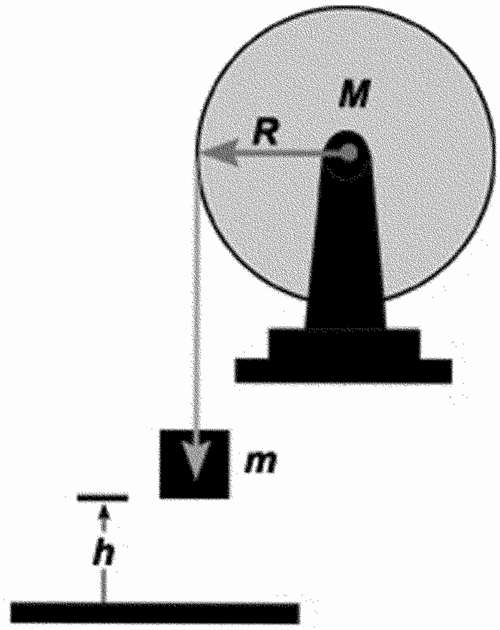
By symmetry

$$I_x = I_y.$$

$$2 I_x = I_z = \frac{1}{6} M a^2$$

Example

A light rope is wrapped around a solid cylinder of mass M and radius R.
Mass m tied to rope and released a height h above the floor. Assuming frictionless motion, what is speed of m and angular velocity of cylinder when m strikes floor?



As the cylinder rotates, the rope unwinds and mass *m* drops.

Ans: System initially has no kinetic energy but has PE.
Finally both m and M have KE and the PE of m is decreased.

$$E_1 = K_1 + U_1$$
$$= 0 + mgh$$

$$E_2 = K_2 + U_2$$
$$= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0$$

$v = R\omega$ [Related by geometry - see Kinematics]

$I = \frac{1}{2}MR^2$ [Solid cylinder]

$E_1 = E_2$ conservation of energy

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\tfrac{1}{2}MR^2\right)(v/R)^2 = \tfrac{1}{2}(m + M/2)v^2$$

$$v = \sqrt{2gh/(1+M/2m)}$$

Angular Momentum of a Particle

- For a particle, linear momentum is not conserved if a force acts on it.

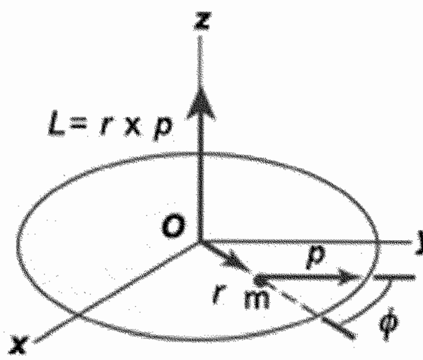
$$\frac{d\vec{P}}{dt} = \vec{F} \qquad \text{(Newton's 2'nd Law)}$$

- For some special types of forces the particle angular momentum is conserved. These forces are called central forces

$$\vec{F} = f\,\hat{r}$$

Consider a particle of mass m, located at the position vector $\vec{r}$, and moving with velocity $\vec{v}$.

The instantaneous angular momentum $\vec{L}$ of the particle relative to the origin 'O' is defined by the cross product of its instantaneous position vector and its instantaneous linear momentum $\vec{p}$:

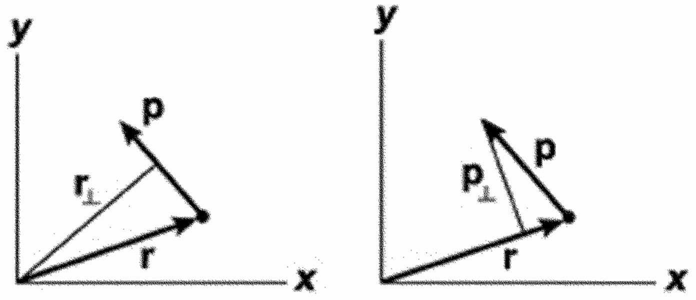


The angular momentum *L* of a particle of mass *m* and momentum *p* located at the position *r* is a vector given by *L = r x p*. Note that the value of L depends on the origin and is a vector perpendicular to both *r* and *p*.

$$\vec{L} = \vec{r} \times \vec{p} = rp\sin\phi$$

$$[L] = kg \cdot m/s^2$$

- First use of vector cross-product
- Magnitude and direction of L depend on choice of coordinate system
- Direction of $\vec{L}$ is $\perp$ to plane containing $\vec{r}$ and $\vec{p}$. Lies on RH normal to this plane.
- $\vec{L} \equiv 0$ if $\vec{r}$ parallel to $\vec{p}$.



$L_z = r_\perp p$ $\quad\quad$ $L_z = rp_\perp$

Geometrically:

$r_\perp \equiv$ perpindicular distance between the origin and the line of $\vec{p}$.

$p_\perp \equiv$ component of $\vec{p} \perp \vec{r}$.

$\vec{L} = \vec{r} \times \vec{p}$ $\quad\quad$ $L_z = r_\perp p = r p_\perp$

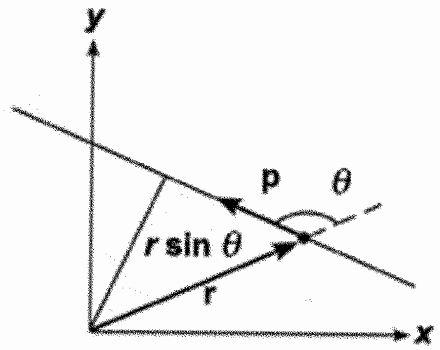
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & r_z \\ p_x & p_y & p_z \end{vmatrix}$$

i) Particle moving in a Straight Line

- Assume $\vec{F} = 0$, $\vec{v} = \text{const.}$
- Direction of $\vec{L} = \text{const.}$
- Magnitude of $\vec{L} = \text{const.}$

$$\vec{L} = rp\sin\theta \ \hat{k}$$

If $\vec{r}$ and $\vec{p}$ are in xy-plane, $\vec{L}$ is along $\hat{k}$



The distance between the origin and the line of motion is $r\sin\theta$.

A particle will have non-zero angular momentum about some origin if its position vector measured from this point appears to rotate about this point.

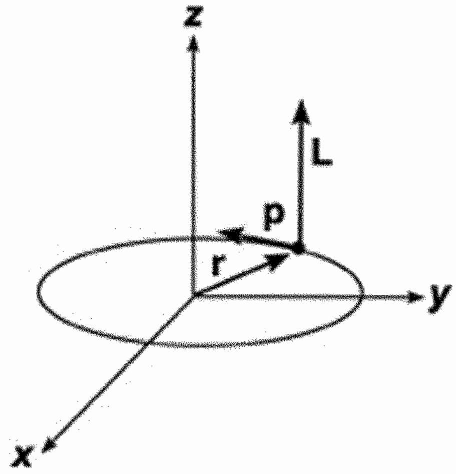
If the position vector only increases or decreases in magnitude, the particle is moving along a line passing through the origin and it therefore has zero angular momentum wrt this origin.

"Choice of coordinate system is crucial. Must define an origin befor calculating angular momentum".

ii) <u>Particle in uniform circular motion</u>.

- ball at the end of a string.
- earth and planets around sun.

- choose origin at centre of circle.



A particle in uniform circular motion. The vector L is perpendicular to the plane of the circle.

$$\vec{L} = \vec{r} \times \vec{p}$$

$\perp$ $\vec{p} \neq$ constant.

$$\vec{L} = rp\,\hat{z} = mrv\,\hat{z} = (mr^2)\omega\,\hat{z} = I\omega\,\hat{k}$$

magnitude is constant since r and p are always $\perp$.

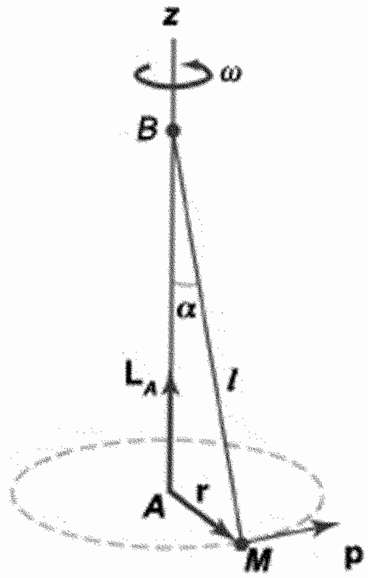
Direction is $\perp$ to plane of circle and is fixed.
L remains constant in <u>magnitude</u> and <u>direction</u>.

In this example there <u>are</u> forces acting.

$$a_c = \frac{mv^2}{r} \qquad \text{and} \qquad F_c = \frac{mv^2}{r}$$

Force acts towards center of circle (origin).
If we had chosen some other origin, $\vec{L}$ would not have been a constant !!!

Angular Momentum of a Conical Pendulum



- Assume circular motion with constant $\underline{\omega}$.

(a)

- choose origin at A :

$$\vec{L}_A = \vec{r} \times \vec{p}$$

$$= r p \, \hat{k}$$

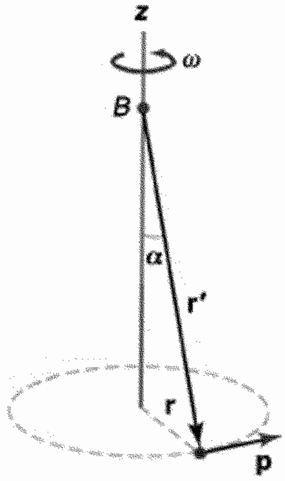
r = radius of circle

$$p = mv = mr\omega$$

$$\vec{L}_A = m r^2 \omega \, \hat{k}$$

$\vec{L}_A$: constant in magnitude
constant in direction

(b) chose origin at B at the pivot:



$$|L_B| = |\vec{r}\,' \times \vec{p}\,|$$

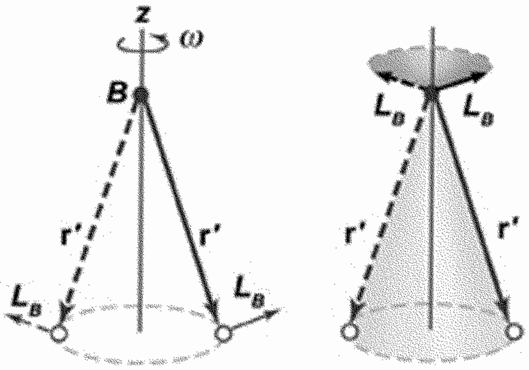
$$= |\vec{r}\,'|\,|\vec{p}\,|$$

$$|L_B| = M v L = M r L \omega$$

$|\vec{r}\,'| = L$; length of string

$\vec{L}_B$: magnitude not constant, depends on location B.
direction of $\vec{L}_B$ is also not constant.

- For fixed B, magnitude is constant.
- Direction sweeps through a cone for each rotation.



- z-component of $\vec{L}_B$ is constant.
- Horizontal component travels in a circle with angular velocity ω.
- Dynamical consequences !!