

$$\vec{a} = \vec{a}_r + \vec{a}_t \quad |\vec{a}| = R\sqrt{\alpha^2 + \omega^4}$$

Rotational Kinetic Energy

$$K = \frac{1}{2} \sum_i m_i v_i^2$$

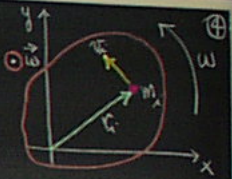
$$v_i = r_i \omega$$

$$K = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$

$$K = \frac{1}{2} I \omega^2$$

$$I = \sum_i m_i r_i^2$$

- Moment of Inertia
- Depends on axis of rotation.
- Particles at large r contribute most!



$\alpha, M \longleftrightarrow$ Reqs. to Linear Motion
 $\alpha, I \longleftrightarrow$ Reqs. to Rotational Motion

} Inertial Quantities

Example: 4 Rotating Masses

$\omega = \omega_0$: ang velocity constant.
 $\alpha = 0$

a) Rotate about y-axis:

$$I_y = \sum m_i r_i^2 = Ma^2 + Ma^2 + 0 + 0 = 2Ma^2$$

$$K_y = \frac{1}{2} I_y \omega^2 = \frac{1}{2} (2Ma^2) \omega^2$$

b) Rotate about z-axis:

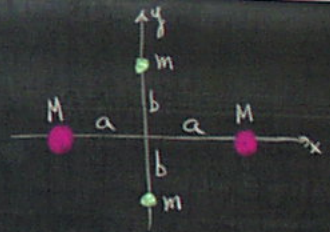
$$I_z = \sum m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2$$

$$= 2Ma^2 + 2mb^2$$

$$K_z = \frac{1}{2} I_z \omega_0^2 = \frac{1}{2} (2(Ma^2 + mb^2)) \omega_0^2$$

$$I_z > I_y$$

$$K_z > K_y$$



Example 4 Rotating Masses

$\omega = \omega_0$: ang velocity constant.
 $\alpha = 0$

a) Rotate about y-axis:

$$I_y = \sum m_i r_i^2 = Ma^2 + Ma^2 + 0 + 0 = 2Ma^2$$

$$K_y = \frac{1}{2} I_y \omega^2 = \frac{1}{2} (2Ma^2) \omega_0^2$$

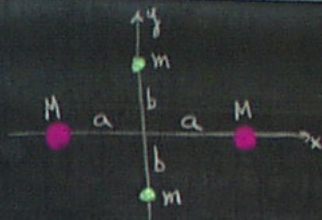
b) Rotate about z-axis:

$$I_z = \sum m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2 = 2Ma^2 + 2mb^2$$

$$K_z = \frac{1}{2} I_z \omega_0^2 = \frac{1}{2} (2(Ma^2 + mb^2)) \omega_0^2$$

$$I_z > I_y$$

$$K_z > K_y$$



Rigid Bodies: Moments of Inertia:

$K = \frac{1}{2} I \omega^2$ KE for rotation about fixed axis.

Δm = element of mass.

$I = \sum r_i^2 \Delta m$ about rot. axis.

$$I = \lim_{\Delta m \rightarrow 0} \sum r_i^2 \Delta m = \int r^2 dm$$

let $\rho(x, y, z)$ = local density (mass/volume)

$$\rho = \lim_{\Delta V \rightarrow 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

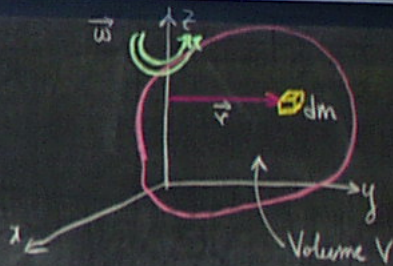
$$\therefore dm = \rho dV$$

$$I = \int_V \rho r^2 dV$$

$I \Rightarrow$ 2nd moment of mass dist.

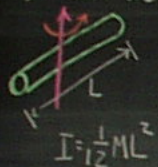
$$\text{If } \rho(x, y, z) = \text{constant} \Rightarrow I = \frac{M}{V} \int_V r^2 dV$$

$$\text{where } \rho \equiv \frac{M}{V}$$

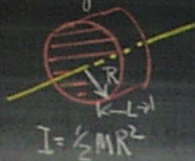


Examples: Moments of Inertia.

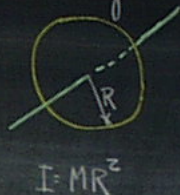
a) Thin Rod



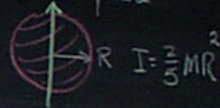
b) Solid Disk Cylinder



c) Ring

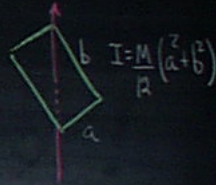


d) Solid Sphere



e) Hollow Sphere/Shell
 $I = \frac{2}{3} MR^2$

f) Plate



Example: Uniform-Hollow Cylinder

Outer Radius R_2

Inner Radius R_1

Axis along center.

Choose cylindrical shell:

Radius r , Thickness dr

length L

$\rho = \text{const.}$ homogeneous/uniform

Volume of Shell $dV = 2\pi r L (dr)$

Mass of Shell $= dm = \rho dV$
 $= 2\pi r \rho L r (dr)$

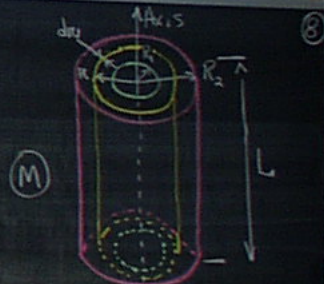
$$I = \int r^2 dm = 2\pi \rho L \int_{R_1}^{R_2} r^3 dr$$

$$I = \frac{\pi \rho L}{2} (R_2^4 - R_1^4)$$

$$= \frac{\pi \rho L}{2} (R_2^2 - R_1^2)(R_2^2 + R_1^2)$$

Mass of Cyl $M = \rho V = \pi L \rho (R_2^2 - R_1^2)$

$$I = \frac{M}{2} (R_2^2 + R_1^2)$$



Example Uniform-Hollow Cylinder

Outer Radius R_2

Inner Radius R_1

Axis along center

Choose cylindrical shell:

Radius r , Thickness dr

length L

$\rho = \text{const.}$ homogeneous/uniform

Volume of Shell $dV = 2\pi r L (dr)$

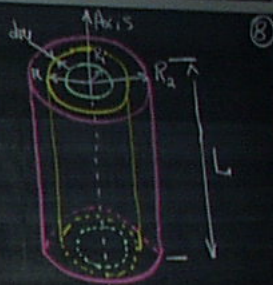
Mass of Shell $dm = \rho dV$
 $= 2\pi \rho L r (dr)$

$$I = \int r^2 dm = 2\pi \rho L \int_{R_1}^{R_2} r^3 dr$$

$$I = \frac{\pi \rho L}{2} (R_2^4 - R_1^4)$$
$$= \frac{\pi \rho L}{2} (R_2^2 - R_1^2)(R_2^2 + R_1^2)$$

Mass of Cyl $M = \rho V = \pi L \rho (R_2^2 - R_1^2)$

$$\therefore I = \frac{M}{2} (R_2^2 + R_1^2)$$



If cylinder is solid: $R_1 = 0$

$$I = \frac{1}{2} MR^2$$

If cylinder is a thin shell:

$R_1 \approx R_2 \approx R$

$$I = MR^2$$

Radius of Gyration: k

$$I = Mk^2$$

Same I as for a mass M located at a 'radius' = " k "

Uniform Thin Rod.

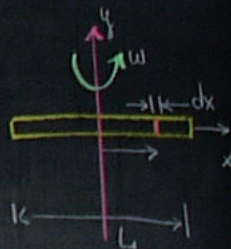
length = L

Mass = M

Slice dx has mass dm

$$dm = M \left(\frac{dx}{L} \right)$$

$$I_y = \int r^2 dm$$
$$= \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx$$
$$= \frac{M}{L} \left. \frac{x^3}{3} \right|_{-L/2}^{L/2} = \frac{ML^2}{12}$$



Rotational Kinetic Energy

Consider a rigid body as a collection of small particles. The KE of a rotating rigid body is the sum of the individual KE's of all the particles.

$$K = \sum_{i=1}^N \frac{1}{2} m_i v_i^2$$

Suppose the rigid body is rotating about a fixed z-axis with an angular velocity ω . All particles execute circular motion with same angular speed.

$$v_i = r_i \omega$$

$$K = \sum_{i=1}^n \frac{1}{2} m_i r_i^2 \omega^2$$

$$K = \frac{1}{2} I \omega^2$$

where $I = \sum_i m_i r_i^2$

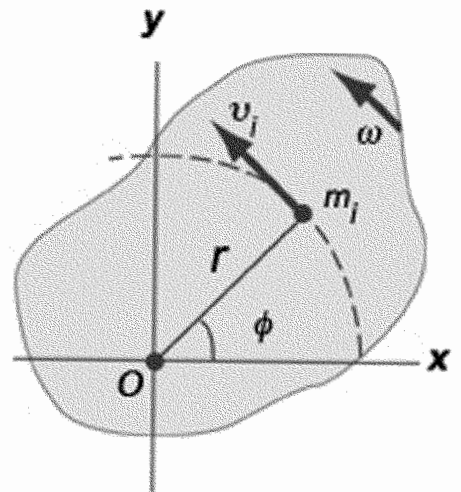
[Moment-of-Inertia]

↑ Particles at large r_i have higher speed and contribute more to KE

$$[I] = \text{kg} \cdot \text{m}^2 \quad (\text{SI})$$

$$\text{slug} \cdot \text{ft}^2 \quad (\text{Br})$$

$\omega, I \leftarrow$ Resistance to rotational motion } Inertial quant.
 $v, m \leftarrow$ Resistance to linear motion }

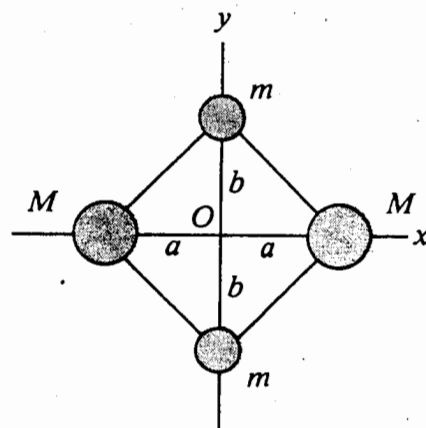


A rigid body rotating about the z axis with angular velocity ω . The kinetic energy of the particle of mass m_i is $\frac{1}{2} m_i v_i^2$. The total kinetic energy of the body is $\frac{1}{2} I \omega^2$.

$r_i =$ particle distance from axis of rot.

Example : Four Rotating Particles

- Four point masses fastened to a very light frame lying in xy -plane.



a) Rotation about y -axis with ang. velocity ω .

• masses m do not contribute since $r_i = 0$ for them and they have no motion about y !

$$I_y = \sum m_i r_i^2 = Ma^2 + Ma^2 = 2Ma^2$$

$$K = \frac{1}{2} I_y \omega^2 = \frac{1}{2} (2Ma^2) \omega^2 = Ma^2 \omega^2$$

b) Rotation about z -axis, \perp to xy -plane

r_i , in each case is the \perp distance to axis of rot.

$$\begin{aligned} I_z &= \sum m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2 \\ &= 2Ma^2 + 2mb^2 \end{aligned}$$

$$K = \frac{1}{2} I_z \omega^2 = (Ma^2 + mb^2) \omega^2$$

Summary:

- Moment of Inertia depends on axis of rotation.
- It will take more work, for this example, to set the system into rotation about z -axis than about y -axis. Depends on the distribution of mass.

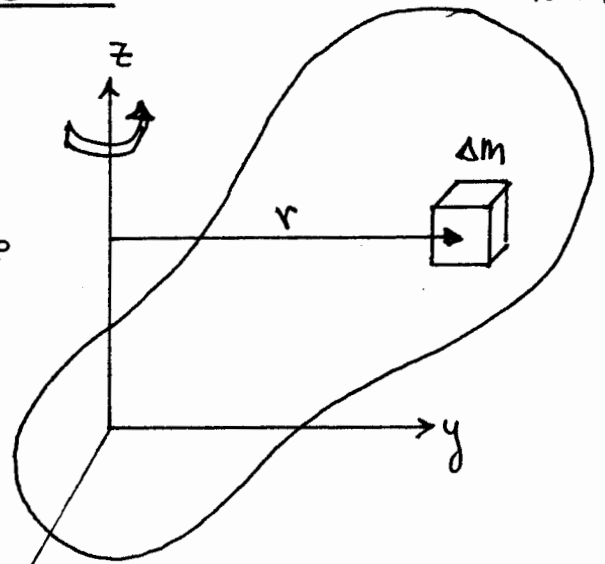
Moments of Inertia for Rigid Bodies

22-12

We evaluate I , for a rigid body rotating about a fixed axis by dividing it up into volume elements of mass Δm .

Use $I = \sum r^2 \Delta m$ and take the limit of this sum as $\Delta m \rightarrow 0$ we have an integral over the volume.

r : \perp distance from rotation axis to Δm .



$$I = \lim_{\Delta m \rightarrow 0} \sum r^2 \Delta m = \int_V r^2 dm$$

dm : must be expressed in terms of its coordinates.

- For a 3-dimensional object it is convenient to do this in terms of the local volume density, i.e. mass per unit volume

$$\rho = \lim_{\Delta V \rightarrow 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

$$\therefore dm = \rho dV$$

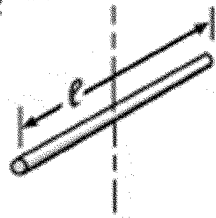
and

$$\boxed{I = \int \rho r^2 dV}$$

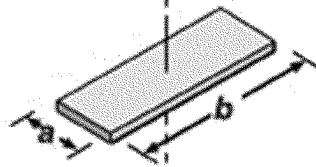
↑
2nd moment of mass distribution.

$$I = \frac{M}{\rho} \int r^2 dV, \text{ for homogeneous bodies}$$

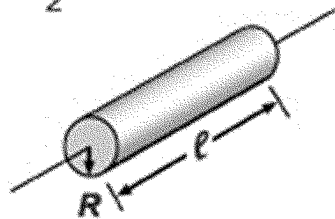
a) Thin rod
 $I = \frac{1}{12} M \ell^2$



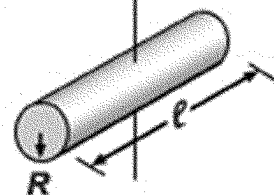
b) Rectangular plate
 $I = \frac{1}{12} M (a^2 + b^2)$



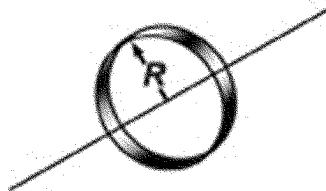
c) Solid cylinder
 $I = \frac{1}{2} MR^2$



d) Solid cylinder
 $I = \frac{1}{4} MR^2 + \frac{1}{12} M \ell^2$



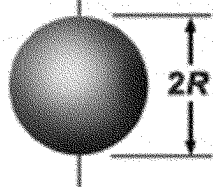
e) Thin-walled cylinder or ring
 $I = MR^2$



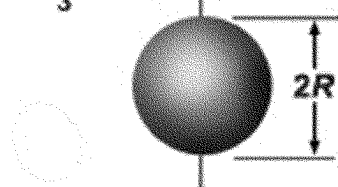
f) Thin-walled cylinder or ring
 $I = \frac{1}{2} MR^2$



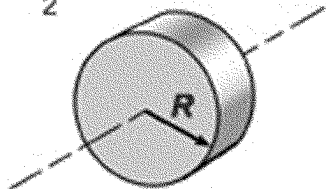
g) Solid sphere
 $I = \frac{2}{5} MR^2$



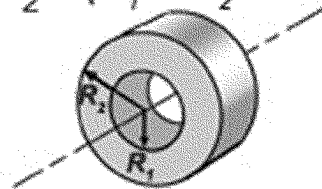
h) Hollow spherical shell
 $I = \frac{2}{3} MR^2$



i) Solid disc
 $I = \frac{1}{2} MR^2$



j) Annular disc or cylinder
 $I = \frac{1}{2} M (R_1^2 + R_2^2)$



Rational Inertia values for various objects for the indicated axes

Example: I - Uniform Hollow Cylinder

22-14

- Always need to choose mass elements which are a fixed radius from axis of rotation

We choose a thin cylindrical shell of radius r , thickness dr , and length l .

The volume of such a shell is that of a flat sheet of length l , thickness dr and width $2\pi r$.

$$dV = 2\pi l r dr$$

$$dm = \rho dV = 2\pi \rho l r dr$$

$$I = \int r^2 dm = 2\pi \rho l \int_{R_1}^{R_2} r^3 dr$$

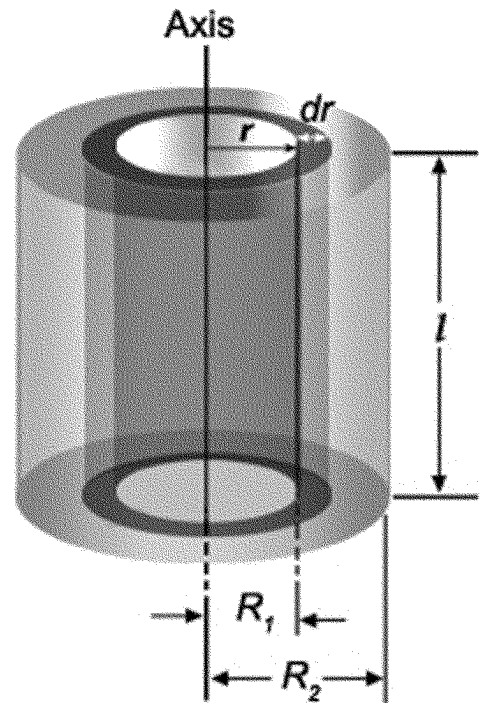
$$= \frac{\pi \rho l}{2} (R_2^4 - R_1^4)$$

$$= \frac{\pi \rho l}{2} (R_2^2 - R_1^2)(R_2^2 + R_1^2)$$

Mass of Cylinder

$$M = \rho V = \pi l \rho (R_2^2 - R_1^2)$$

$$I = \frac{M}{2} (R_2^2 + R_1^2)$$



Moment of inertia of a hollow cylinder. The mass element is a cylindrical shell of a radius r and thickness dr .

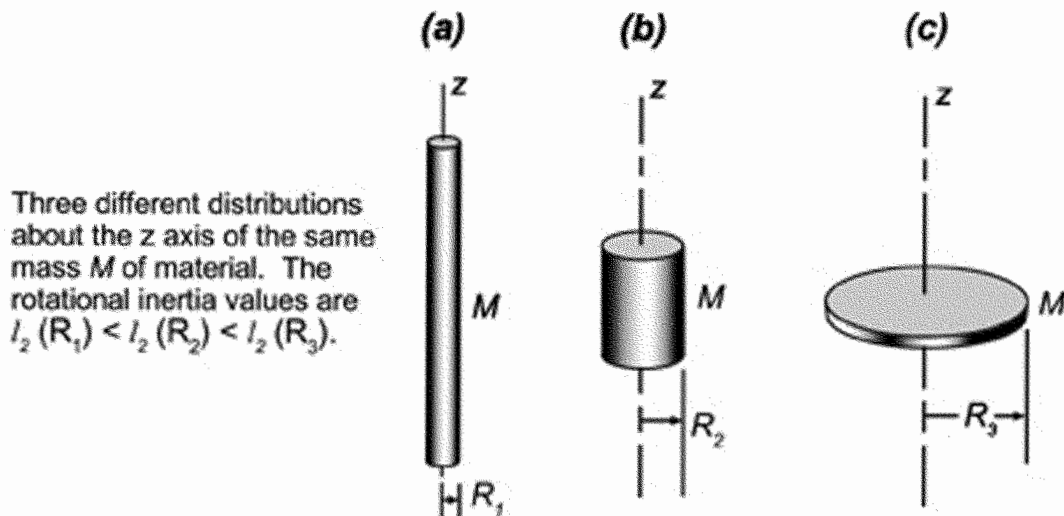
If cylinder is solid, $R_1 = 0$.

$$I = \frac{1}{2} MR^2$$

If cylinder is a very thin shell, $R_1 \sim R_2 = R$

$$I = MR^2$$

I in all cases does not depend on l . The distribution along the axis does not matter. Moment-of-inertia depends on radial distribution.



When using I , it is often convenient to do so in terms of a "Radius of Gyration", k

$$I = M k^2$$

It is defined such that if all of the mass of an object were located a distance k from the axis, it would have the same moment-of-inertia as the actual object.