

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 TEAL

Fall Term 2004

Exam 3: Equation Summary

Momentum:

$$\vec{p} = m\vec{v}, \quad \vec{F}_{ave}\Delta t = \Delta\vec{p}, \quad \vec{F}_{ext}^{total} = \frac{d\vec{p}^{total}}{dt}$$

$$\text{Impulse: } \vec{I} \equiv \int_{t=0}^{t=t_f} \vec{F}(t)dt = \Delta\vec{p}$$

$$\text{Torque: } \vec{\tau}_S = \vec{r}_{S,P} \times \vec{F}_P \quad |\vec{\tau}_S| = |\vec{r}_{S,P}| |\vec{F}_P| \sin\theta = r_{\perp} F = r F_{\perp}$$

Static Equilibrium:

$$\vec{F}_{total} = \vec{F}_1 + \vec{F}_2 + \dots = \vec{0}; \quad \vec{\tau}_S^{total} = \vec{\tau}_{S,1} + \vec{\tau}_{S,2} + \dots = \vec{0}.$$

$$\text{Rotational dynamics: } \vec{\tau}_S^{total} = \frac{d\vec{L}_S}{dt}$$

$$\text{Angular Velocity: } \vec{\omega} = (d\theta/dt)\hat{k}$$

$$\text{Angular Acceleration: } \vec{\alpha} = (d^2\theta/dt^2)\hat{k}$$

$$\text{Fixed Axis Rotation: } \vec{\tau}_S = I_S \vec{\alpha}$$

$$\tau_S^{total} = I_S \alpha = I_S \frac{d\omega}{dt}$$

$$\text{Moment of Inertia: } I_S = \int_{body} dm(r_{\perp})^2$$

$$\text{Angular Momentum: } \vec{L}_S = \vec{r}_{S,m} \times m\vec{v},$$

$$|\vec{L}_S| = |\vec{r}_{S,m}| |m\vec{v}| \sin\theta = r_{\perp} p = r p_{\perp}$$

Angular Impulse:

$$\vec{J}_S = \int_{t_0}^{t_f} \vec{\tau}_S dt = \Delta\vec{L}_S = \vec{L}_{S,f} - \vec{L}_{S,0}$$

Rotation and Translation:

$$\vec{L}_S^{total} = \vec{L}_S^{orbital} + \vec{L}_{cm}^{spin},$$

$$\vec{L}_S^{orbital} = \vec{r}_{S,cm} \times \vec{p}^{total},$$

$$\vec{L}_{cm}^{spin} = I_{cm} \vec{\omega}_{spin}$$

$$\vec{\tau}_S^{orbit} = \frac{d\vec{L}_S^{orbit}}{dt}, \quad \vec{\tau}_{cm}^{spin} = \frac{d\vec{L}_{cm}^{spin}}{dt}$$

Rotational Energy: $K_{cm} = \frac{1}{2} I_{cm} \omega_{cm}^2$

Rotational Power: $P_{rot} \equiv \frac{dW_{rot}}{dt} = \vec{\tau}_s \cdot \vec{\omega} = \tau_s \omega = \tau_s \frac{d\theta}{dt}$

One Dimensional Kinematics: $\vec{v} = d\vec{r} / dt$, $\vec{a} = d\vec{v} / dt$

$$v_x(t) - v_{x,0} = \int_{t'=0}^{t'=t} a_x(t') dt' \quad x(t) - x_0 = \int_{t'=0}^{t'=t} v_x(t') dt'$$

Constant Acceleration:

$$x(t) = x_0 + v_{x,0}(t - t_0) + \frac{1}{2} a_x (t - t_0)^2 \quad v_x(t) = v_{x,0} + a_x (t - t_0)$$

$$y(t) = y_0 + v_{y,0}(t - t_0) + \frac{1}{2} a_y (t - t_0)^2 \quad v_y(t) = v_{y,0} + a_y (t - t_0)$$

where $x_0, v_{x,0}, y_0, v_{y,0}$ are the initial position and velocities components at $t = t_0$

Newton's Second Law: Force, Mass, Acceleration

$$\vec{F} \equiv m\vec{a} \quad \vec{F}^{total} = \vec{F}_1 + \vec{F}_2 \quad F_x^{total} = ma_x \quad F_y^{total} = ma_y \quad F_z^{total} = ma_z$$

Newton's Third Law: $\vec{F}_{1,2} = -\vec{F}_{2,1}$

Force Laws:

Universal Law of Gravity: $\vec{F}_{1,2} = -G \frac{m_1 m_2}{r_{1,2}^2} \hat{r}_{1,2}$, attractive

Gravity near surface of earth: $\vec{F}_{grav} = m_{grav} \vec{g}$, towards earth

Contact force: $\vec{F}_{contact} = \vec{N} + \vec{f}$, depends on applied forces

Static Friction: $0 \leq f_s \leq f_{s,max} = \mu_s N$ direction depends on applied forces

Kinetic Friction: $f_k = \mu_k N$ opposes motion

Hooke's Law: $F = k|\Delta x|$, restoring

Kinematics Circular Motion: arc length: $s = R\theta$; angular velocity: $\omega = d\theta/dt$
 tangential velocity: $v = R\omega$; angular acceleration: $\alpha = d\omega/dt = d^2\theta/dt^2$; tangential acceleration $a_\theta = R\alpha$.

Period: $T = \frac{2\pi R}{v} = \frac{2\pi}{\omega}$; **frequency:** $f = \frac{1}{T} = \frac{\omega}{2\pi}$,

Radial Acceleration: $|a_r| = R\omega^2$; $|a_r| = \frac{v^2}{R}$; $|a_r| = 4\pi^2 R f^2$; $|a_r| = \frac{4\pi^2 R}{T^2}$

Center of Mass: $\vec{R}_{cm} = \sum_{i=1}^{i=N} m_i \vec{r}_i / m^{total} \rightarrow \int_{body} dm \vec{r} / m^{total}$;

Velocity of Center of Mass: $\vec{V}_{cm} = \sum_{i=1}^{i=N} m_i \vec{v}_i / m^{total} \rightarrow \int_{body} dm \vec{v} / m^{total}$

Torque: $\vec{\tau}_S = \vec{r}_{S,P} \times \vec{F}_P$ $|\vec{\tau}_S| = |\vec{r}_{S,P}| |\vec{F}_P| \sin \theta = r_{\perp} F = r F_{\perp}$

Static Equilibrium: $\vec{F}_{total} = \vec{F}_1 + \vec{F}_2 + \dots = \vec{0}$; $\vec{\tau}_S^{total} = \vec{\tau}_{S,1} + \vec{\tau}_{S,2} + \dots = \vec{0}$.

Kinetic Energy: $K = \frac{1}{2}mv^2$; $\Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2$

Work: $W = \int_{r_0}^{r_f} \vec{F} \cdot d\vec{r}$; **Work- Kinetic Energy:** $W^{total} = \Delta K$

Power: $P = \vec{F} \cdot \vec{v} = dK/dt$

Potential Energy: $\Delta U = -W_{conservative} = -\int_A^B \vec{F}_c \cdot d\vec{r}$

Potential Energy Functions with Zero Points:

Constant Gravity: $U(y) = mgy$ with $U(y_0 = 0) = 0$.

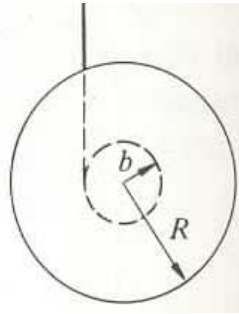
Inverse Square Gravity: $U_{gravity}(r) = -\frac{Gm_1m_2}{r}$ with $U_{gravity}(r_0 = \infty) = 0$.

Hooke's Law: $U_{spring}(x) = \frac{1}{2}kx^2$ with $U_{spring}(x = 0) = 0$.

Work- Mechanical Energy: $W_{nc} = \Delta K + \Delta U^{total} = \Delta E_{mech} = (K_f + U_f^{total}) - (K_0 + U_0^{total})$

Problem 3: Torque, Rotation and Translation

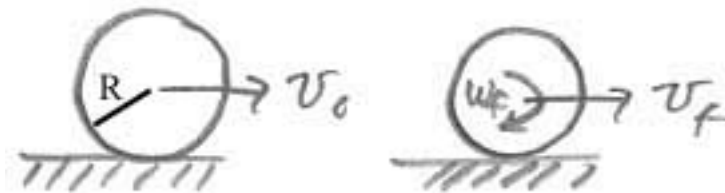
A Yo-Yo of mass m has an axle of radius b and a spool of radius R . Its moment of inertia can be taken to be $I = (1/2)mR^2$ and the thickness of the string can be neglected. The Yo-Yo is released from rest.



- What is the tension in the cord as the Yo-Yo descends?
- Use conservation of energy to find the angular velocity of the Yo-Yo when it reaches the bottom of the string.
- What happens to the Yo-Yo at the bottom of the string?

~~Problem 4: Rolling Cylinder~~

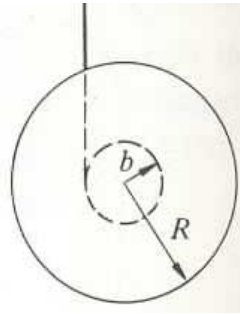
~~A solid cylinder of mass m and radius R is initially thrown along a wooden floor hallway with an initial speed v_0 and zero initial angular velocity $\omega_0 = 0$ as shown in the figure below.~~



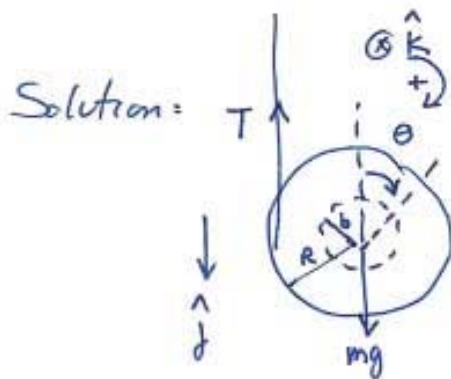
- ~~What is the moment of inertia of the cylinder about the axis of rotation of the cylinder?~~
- ~~What effects does friction have on the cylinder before it reaches its final speed? Is the friction static or kinetic? Which direction does it point?~~
- ~~What is the angular speed, ω_f , of the cylinder when it just starts to roll without slipping? Express your answer in terms of m , R , and v_0 .~~

Problem 3: Torque, Rotation and Translation

A Yo-Yo of mass m has an axle of radius b and a spool of radius R . Its moment of inertia can be taken to be $I = (1/2)mR^2$ and the thickness of the string can be neglected. The Yo-Yo is released from rest.



- a) What is the tension in the cord as the Yo-Yo descends?
- b) Use conservation of energy to find the angular velocity of the Yo-Yo when it reaches the bottom of the string.
- c) What happens to the Yo-Yo at the bottom of the string?



torque and force diagram; with this choice of sense of rotation (clockwise) and positive $\hat{j}$ direction

Then $a = b \alpha$ (1)

Newton's 2nd Law: $mg - T = ma$ (2)

Torque Equation: $bT = I_{cm} \alpha$ (3)

moment of inertia: $I_{cm} = \frac{1}{2} m R^2$ (4)

Use $mg - T = ma$ to solve for T .

$\Rightarrow mg - T = m b \alpha$ eq (1)

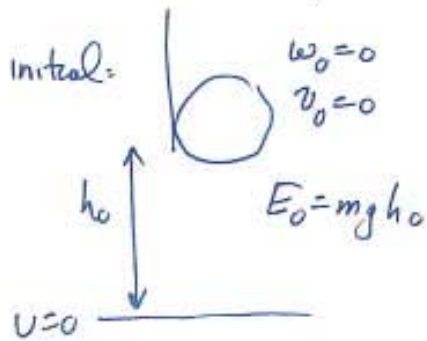
$\Rightarrow mg - T = m \frac{b b T}{I_{cm}}$ eq (3)

$mg = T \left(1 + \frac{m b^2}{I_{cm}} \right)$

$\Rightarrow T = \frac{mg}{1 + \frac{m b^2}{I_{cm}}} = \frac{mg}{1 + \frac{m b^2}{\frac{1}{2} m R^2}}$

note: $\alpha = \frac{bT}{I_{cm}} = \frac{bT}{\frac{1}{2} m R^2} = \frac{b}{\frac{1}{2} m R^2} \frac{mg}{1 + \frac{m b^2}{\frac{1}{2} m R^2}}$
 $a = b \alpha = \frac{b^2 T}{\frac{1}{2} m R^2}$

Conservation of Energy



$$E_f = \frac{1}{2}mv_f^2 + \frac{1}{2}I_{cm}\omega_f^2$$

$$v_f = b\omega_f, I_{cm} = \frac{1}{2}mR^2$$

②

$$E_f = \frac{1}{2}mv_f^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega_f^2$$

$$v_f = \frac{1}{2}mb^2\omega_f^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega_f^2$$

$$E_0 = mgh_0 = E_f = \left(\frac{1}{2}mb^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\right)\omega_f^2$$

$$\Rightarrow \omega_f = \left(\frac{mgh_0}{\left(\frac{1}{2}mb^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\right)} \right)^{1/2}$$

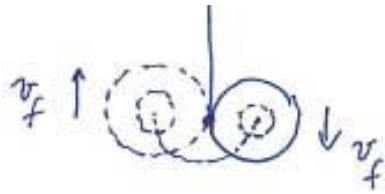
note: $h_0 = \frac{1}{2}at^2 \Rightarrow t = \left(\frac{2h_0}{a}\right)^{1/2}$

$$\omega_f = \alpha t = \alpha \left(\frac{2h_0}{a}\right)^{1/2}$$

$$= \left(\frac{2h_0 \frac{1}{2}mR^2}{\frac{1}{2}mR^2} \right)^{1/2}$$

$$\omega_f = \left(\frac{2h_0 \frac{1}{2}mR^2}{\frac{1}{2}mR^2} \frac{mg}{1 + \frac{mb^2}{\frac{1}{2}mR^2}} \right)^{1/2}$$

c) At the bottom of the descent, the Yo-Yo is spinning with angular velocity ω_f . The Yo-Yo pivots about the end of the string.



with angular velocity $\omega_f = \frac{d\theta}{dt} = \frac{\pi \text{ rad}}{\Delta t}$



where Δt is the time it takes the Yo-Yo to go π radians. The momentum has reversed so,

$$\vec{I} = \Delta \vec{p}$$

$$\text{the impulse } \vec{I} = \int_0^{\Delta t} (T + mg) \hat{j} dt = -m v_f \hat{j} - m v_f \hat{j}$$

$$\Rightarrow \Delta t(mg - T) = -2m v_f$$

$$\text{or } T = mg + 2 \frac{m v_f}{\Delta t} = mg + 2 \frac{m v_f \omega_f}{\pi}$$

Since $v_f = b \omega_f$ we have

$$T = mg + 2 \frac{m b}{\pi} \omega_f^2$$