

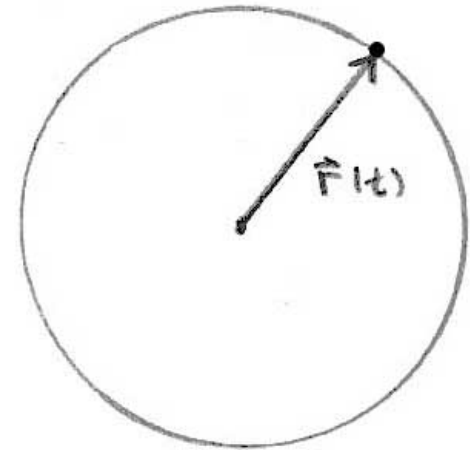
Circular Motion

8.01T

Sept 27, 2004

Position and Displacement

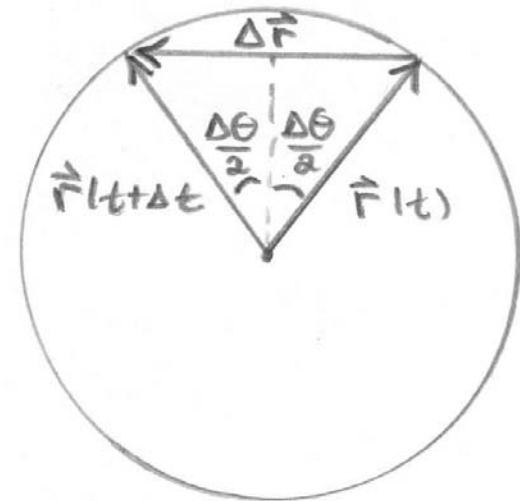
Position vector of an object moving in a circular orbit of radius R



Change in position $\Delta\vec{r}$ between time t and time $t+\Delta t$.

$$\rightarrow |\vec{r}| \sin \frac{\Delta\theta}{2} \leftarrow$$

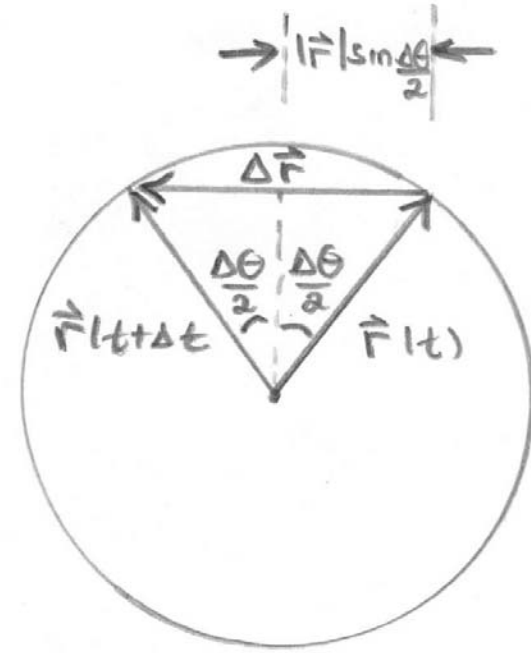
Position vector is changing in direction not in magnitude.



Magnitude of Displacement

- The magnitude of the displacement, is the length of the chord of the circle

$$|\Delta \vec{r}| = 2R \sin(\Delta \theta / 2)$$



Small Angle Approximation

- When the angle is small, approximate

$$\sin \Delta \theta \cong \Delta \theta$$

- infinite power series expansion

$$\sin \Delta \theta = \Delta \theta - \frac{1}{3!}(\Delta \theta)^3 + \frac{1}{5!}(\Delta \theta)^5 - \dots$$

- Using the small angle approximation, the magnitude of the displacement is

$$|\Delta \vec{r}| \cong R \Delta \theta$$

Magnitude of Velocity and Angular Velocity

Magnitude of the velocity is proportional to the rate of change of the magnitude of the angle with respect to time

$$v \equiv |\vec{v}| = \lim_{\Delta t \rightarrow 0} \frac{|\Delta \vec{r}|}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{R|\Delta \theta|}{\Delta t} = R \lim_{\Delta t \rightarrow 0} \frac{|\Delta \theta|}{\Delta t} = R \left| \frac{d\theta}{dt} \right|$$

angular velocity

$$\omega \equiv \frac{d\theta}{dt}$$

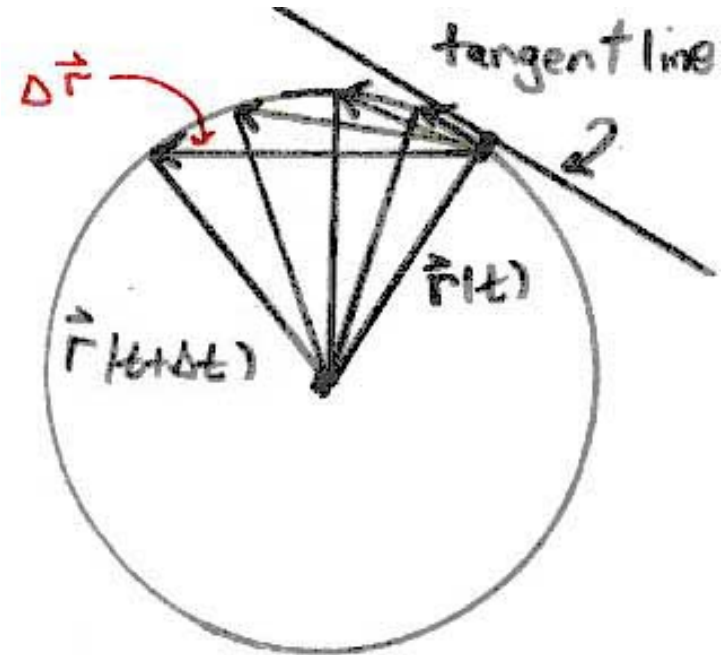
units: [rad-sec⁻¹]

magnitude of velocity

$$v = R\omega$$

Direction of Velocity

- sequence of chord $\Delta \vec{r}$ directions as Δt approaches zero
- the direction of the velocity at time t is perpendicular to position vector and tangent to the circular orbit



Acceleration

When an object moves in a circular orbit, the acceleration has two components, tangential and radial.

Tangential Acceleration

the *tangential acceleration* is just the rate of change of the magnitude of the velocity

$$a_{\theta} = \lim_{\Delta t \rightarrow 0} \frac{\Delta v_{\theta}}{\Delta t} = R \lim_{\Delta t \rightarrow 0} \frac{\Delta \omega}{\Delta t} = R \frac{d\omega}{dt} = R \frac{d^2\theta}{dt^2}$$

Angular acceleration: rate of change of angular velocity with time

$$\alpha \equiv \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

units are $\left[\text{rad} \cdot \text{sec}^{-2} \right]$

tangential acceleration $a_{\text{tan}} = R\alpha$

Uniform Circular Motion

- object is constrained to move in a circle and total tangential force acting on the object is zero, then by Newton's Second Law, the tangential acceleration is zero

$$a_{\text{tan}} = 0$$

- magnitude of the velocity (speed) remains constant

Period and Frequency

- The amount of time to complete one circular orbit of radius R is called the period.
- In one period the object travels a distance equal to the circumference,

$$s = 2\pi R = vT$$

- Period:

$$T = 2\pi R / v = 2\pi R / R\omega = 2\pi / \omega$$

- Frequency is the inverse of the period

$$f = 1/T = \omega / 2\pi$$

- Units $[\text{sec}^{-1}] \equiv [\text{Hz}]$