

Problem 3: Experiment 7: Collisions Analysis

Part One Inelastic Collision:

Analysis: Complete the analysis of your data table by following the two steps below, and answer Question 1 below.

You will analyze this collision as seen by two different observers. One observer is at rest with respect to the classroom. This reference frame is called the lab reference frame. The observer is moving at a velocity equal to the center of mass velocity of the two cart system. This frame is called the center of mass reference frame.

Step 1: Draw two momentum diagrams for both carts immediately before and immediately after the collision, one describing the collision in the lab reference frame, and one describing the collision in the center of mass reference frame.

Step 2: You made three different trial collisions; the first with equal masses, and the second and third with unequal masses. Use your measured values to complete the following table. Entries to the left of the double vertical bar are before the two carts collide; those to the right are after the two carts collide but before the collision with the force sensor. Hand the table in with your homework solutions.

m_A	m_B	v_{A1}	v_{CM}	K_1	K_{CMCS}	v_2	K_2	W_{NC}
0.25 kg	0.25 kg							
0.25 kg	0.50 kg							
0.50 kg	0.25 kg							

a) First calculate the initial kinetic energy K_1 and the final kinetic energy K_2 in the 'lab reference frame'. The difference between the two, $K_2 - K_1 = W_{nc}$, is the non-conservative work internal work, $W_{nc} = K_f - K_0$, due to the totally inelastic collision. This work is due to forces that irreversibly deform the bodies during the collision.

b) Then calculate the velocity of the center of mass which is given by

$$\vec{v}_{cm} = \frac{m_A \vec{v}_{A,1}}{m_A + m_b}.$$

Because momentum is constant during the collision, you should expect $\vec{v}_{cm} = \vec{v}_2$. Calculate the total kinetic energy of the two carts in the center of mass reference frame, given by

$$K_{CMCS} = \frac{1}{2} m_A (v_{A,1} - v_{cm})^2 + \frac{1}{2} m_B (v_{cm})^2$$

Question 1: In the center of mass reference frame: the velocity of the incident cart A moves forward with a slower velocity, $\vec{v}_{A,1} - \vec{v}_{cm}$, than in the lab frame; the cart B moves backward with a velocity, $-\vec{v}_{cm}$; After the collision, the two carts are stuck together and are at rest! So the final kinetic energy is zero, $K_f' = 0$. Therefore the non-conservative work in the center of mass reference frame is given by

$$W_{nc} = K_f' - K_{CMCS} = -K_{CMCS}.$$

Show that $K_f - K_0 = -K_{cmcs}$. This will require a calculation. Does your experimental data verify this theoretical result?

Part Two Elastic Collision:

Analysis: Complete the analysis of your data table by following the two steps below, and answer Question 2 below.

Once again you made three trials. Since the motion sensor can only calculate the initial and final velocities of the incident cart, you needed to measure the impulse of the second cart on the force sensor to determine the velocity of the target cart.

Step 1: Draw a momentum diagram for the collision immediately before the cart B hits the force sensor and after cart B rebounds from the force sensor.

Step 2: If we assume that cart B collides elastically with the force sensor, the momentum of the cart B changes by $\Delta \vec{p}_B = -2m_B \vec{v}_{B,2}$. This change in momentum is the impulse that the force sensor exerts on the target cart. The target cart therefore exerts an equal but opposite impulse on the force sensor with magnitude $J = 2m_B v_{B,2}$. So measuring the impulse allows you to calculate the velocity of the target cart, $v_{B,2} = J/2m_B$. Use your measured values to complete the following table. Velocity entries to the left of the double vertical bar are before the two carts collide; velocity entries to the right are after they collide but before the collision with the force sensor. Hand the table in with your homework solutions.

m_A	m_B	v_{A1}	v_{CM}	K_1	v_{A2}	J	v_{B2}	K_2
0.25 kg	0.25 kg							
0.25 kg	0.75 kg							
0.75 kg	0.25 kg							

Question 2: Assume that the collision is elastic. Use the principles of conservation of momentum and energy to calculate the theoretical final velocities for cart B and cart A for the three trials. Show your work. (These calculations are involved for the second and third trials.) Briefly describe whether or not your measured values agree with your theoretical values. Are the collisions elastic?

Part Three: Modeling the Elastic Collision

Question 3: The repulsive force between the two carts is modeled by the force law

$$F = Ae^{-x/l}$$

where x is the separation between the carts and l is a characteristic length associated with the collision. If the carts start very far apart (an infinite distance), the work required to bring them to a separation distance d is

$$W = \int_{\infty}^d F dx = \int_{\infty}^d Ae^{-x/l} dx.$$

Calculate this integral. The result is the equivalent of the potential energy stored in a spring $(1/2)kx^2$ that satisfies a linear restoring force law.

Question 4: Suppose there are two of these carts on a track, one cart with mass m and speed $v_{A,1}$ runs into the other, which has equal mass and is initially at rest. Find the minimum separation d between the carts during the collision. You may assume that A is sufficiently large that so that $d > l$. Evaluate your result numerically for $A = 2.0 \times 10^1$ N, $l = 5.0 \times 10^{-3}$ m, $m = 2.5 \times 10^{-1}$ kg, and $v_{A,1} = 5.0 \times 10^{-1}$ m · s⁻¹. Hint: This problem is most easily done in the center of mass reference frame.

Problem 3: Experiment 9: Collisions

Part One Inelastic Collision:

Analysis: Complete the analysis of your data table by following the two steps below, and answer Question 1 below.

You will analyze this collision as seen by two different observers. One observer is at rest with respect to the classroom. This reference frame is called the lab reference frame. The observer is moving at a velocity equal to the center of mass velocity of the two cart system. This frame is called the center of mass reference frame.

Step 1: Draw two momentum diagrams for both carts immediately before and immediately after the collision, one describing the collision in the lab reference frame, and one describing the collision in the center of mass reference frame.

Answer:

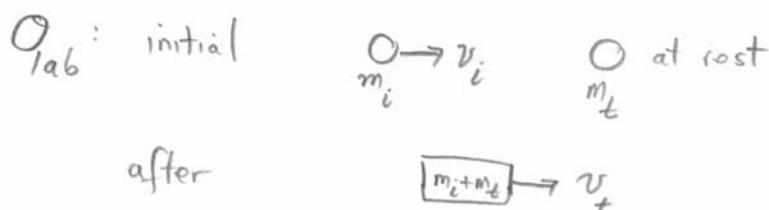


Figure 1: Collision in lab reference frame

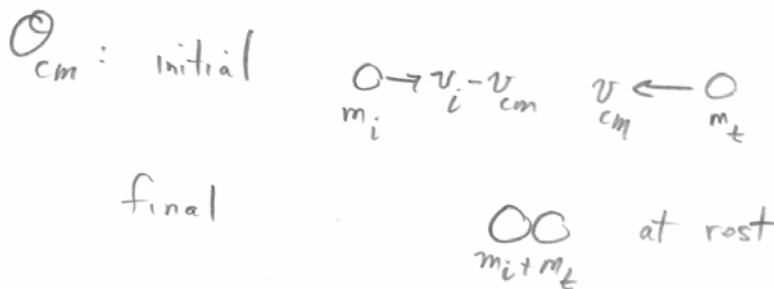


Figure 2: Collision in center of mass reference frame

You made three different trial collisions; the first with equal masses, and the second and third with unequal masses. Use your measured values to complete the following table. Entries to the left of the double vertical bar are before the two carts collide; those to the right are after the two carts collide but before the collision with the force sensor. Hand the table in with your homework solutions.

Step 2:

a) First calculate the initial kinetic energy K_i and the final kinetic energy K_f in the 'lab reference frame'. The difference between the two, $K_f - K_i = W_{nc}$, is the non-conservative work internal work, $W_{nc} = K_f - K_i$, due to the totally inelastic collision. This work is due to forces that irreversibly deform the bodies during the collision.

b) Then calculate the velocity of the center of mass which is given by

$$\vec{v}_{cm} = \frac{m_A \vec{v}_{A,i}}{m_A + m_B}.$$

Because momentum is constant during the collision, you should expect $\vec{v}_{cm} = \vec{v}_2$. Calculate the total kinetic energy of the two carts in the center of mass reference frame, given by

$$K_{CMCS} = \frac{1}{2} m_A (v_{A,i} - v_{cm})^2 + \frac{1}{2} m_B (v_{cm})^2$$

m_i [kg]	m_t [kg]	v_i [m·s ⁻¹]	v_{cm} [m·s ⁻¹]	K_i [J]	K_{cmcs} [J]	v_f [m·s ⁻¹]	K_f [J]	W_{nc} [J]
0.25	0.25	0.60	0.30	0.0450	0.0225	0.30	0.0225	0.0225
0.25	0.50	0.66	0.22	0.0540	0.037	0.21	0.0165	0.038
0.50	0.25	0.43	0.29	0.0462	0.023	0.27	0.0273	0.019

Question 1: In the center of mass reference frame: the velocity of the incident cart A moves forward with a slower velocity, $\vec{v}_{A,i} - \vec{v}_{cm}$, than in the lab frame; the cart B moves backward with a velocity, $-\vec{v}_{cm}$; After the collision, the two carts are stuck together and are at rest! So the final kinetic energy is zero, $K_f' = 0$. Therefore the non-conservative work in the center of mass reference frame is given by

$$W_{nc} = K_f' - K_{CMCS} = -K_{CMCS}.$$

Show that $K_f - K_i = -K_{CMCS}$. This will require a calculation. Does your experimental data verify this theoretical result?

Solution: For the first two cases the data agrees with the theory. For the third case when the incident mass is twice the target mass, the final velocity is less than the center of mass velocity

indicating that there is some additional loss of energy due to friction. Since the target is more massive, the incoming energy loss due to friction may be greater.

In the center of mass reference frame with

$$v_{cm} = \frac{m_i v_i}{m_i + m_t}, \text{ the incident cart has}$$

$$\text{velocity } v_i - v_{cm} = v_i - \frac{m_i v_i}{m_i + m_t} = \frac{m_t v_i}{m_i + m_t}$$

The kinetic energy in the center of mass frame is then

$$(K_{cm,cs})_0 = \frac{1}{2} m_i (v_i - v_{cm})^2 + \frac{1}{2} m_t v_{cm}^2$$

$$= \frac{1}{2} m_i \left(\frac{m_t v_i}{m_i + m_t} \right)^2 + \frac{1}{2} m_t \left(\frac{m_i v_i}{m_i + m_t} \right)^2$$

$$= \frac{1}{2} \frac{m_i m_t (m_i + m_t)}{(m_i + m_t)^2} v_i^2 = \frac{1}{2} \frac{m_i m_t}{m_i + m_t} v_i^2$$

The quantity $\frac{m_i m_t}{m_i + m_t}$ is often called the

reduced mass. Since the two carts are at rest in the center of mass reference frame after the collision, $(K_{cm,cs})_f = 0$

$$W_{nc} = \Delta E = \Delta K_{cm} = -(K_{cm,cs})_0$$

In the lab (ground) reference frame
momentum conservation implies

$$m_i v_i = (m_i + m_t) v_f \Rightarrow v_f = \frac{m_i}{m_i + m_t} v_i = v_{cm}$$

The final carts are moving with the
center of mass velocity. The initial kinetic
energy is

$$K_0 = \frac{1}{2} m_i v_i^2$$

The final kinetic energy is

$$\begin{aligned} K_f &= \frac{1}{2} (m_i + m_t) v_{cm}^2 = \frac{1}{2} (m_i + m_t) \frac{m_i^2 v_i^2}{(m_i + m_t)^2} \\ &= \frac{1}{2} \frac{m_i^2 v_i^2}{m_i + m_t} \end{aligned}$$

So the change in kinetic energy is

$$\begin{aligned} W_{nc} = \Delta K &= K_f - K_0 = \frac{1}{2} \frac{m_i^2 v_i^2}{m_i + m_t} - \frac{1}{2} m_i v_i^2 \\ &= \frac{1}{2} m_i v_i^2 \left(\frac{m_i}{m_i + m_t} - 1 \right) = -\frac{1}{2} \frac{m_i m_t v_i^2}{m_i + m_t} \\ &= -(K_{cm,cs})_0 \end{aligned}$$

Since the carts are constrained to move along the track, there is no rotation in this collision. The displacement of an individual particle during the collision in the lab reference frame can be considered to be a vector sum of the displacement relative to the center of mass, and a displacement due to the center of mass motion,

$$d\vec{r}_{i,lab} = d\vec{r}_{i,cm} + d\vec{r}_{cm}$$

The total internal non-conservative force acting on the particle i due to the interaction with particle j does a work equal to

$$\vec{F}_{i,j} \cdot d\vec{r}_{i,lab} = \vec{F}_{i,j} \cdot d\vec{r}_{i,cm} + \vec{F}_{i,j} \cdot d\vec{r}_{cm}.$$

The total internal non-conservative force acting on the particle j due to the interaction with particle i does a work equal to

$$\vec{F}_{j,i} \cdot d\vec{r}_{j,lab} = \vec{F}_{j,i} \cdot d\vec{r}_{j,cm} + \vec{F}_{j,i} \cdot d\vec{r}_{cm}.$$

When we add these terms together, the second terms cancel,

$$\vec{F}_{i,j} \cdot d\vec{r}_{cm} + \vec{F}_{j,i} \cdot d\vec{r}_{cm} = 0,$$

due to the fact that the forces are equal and opposite by Newton's Third Law,

$$\vec{F}_{i,j} - \vec{F}_{j,i}.$$

Therefore the contribution to the total work is

$$\vec{F}_{i,j} \cdot d\vec{r}_{i,cm} + \vec{F}_{j,i} \cdot d\vec{r}_{j,cm} \neq 0.$$

This is not zero because the individual particles do not necessarily displace the same amount. This is the result of the deformation of the body,

$$d\vec{r}_{i,cm} \neq d\vec{r}_{j,cm}$$

So when we sum over all particle pairs this gives the total internal work. This only depends on the motion relative to the center of mass, so in either reference frame, the internal work is the same, hence the change in kinetic energy should also be the same.

Part Two Elastic Collision:

Analysis: Complete the analysis of your data table by following the two steps below, and answer Question 2-5 below.

Once again you made three trials. Since the motion sensor can only calculate the initial and final velocities of the incident cart, you needed to measure the impulse of the second cart on the force sensor to determine the velocity of the target cart.

Step 1: Draw a momentum diagram for the collision immediately before the cart B hits the force sensor and after cart B rebounds from the force sensor.

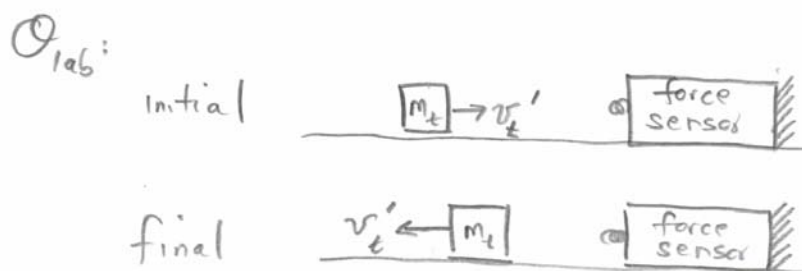


Figure 3: Elastic collision of Cart B with force sensor

Step 2: If we assume that cart B collides elastically with the force sensor, the momentum of the cart B changes by $\Delta \vec{p}_B = -2m_B \vec{v}_{B,2}$. This change in momentum is the impulse that the force sensor exerts on the target cart. The target cart therefore exerts an equal but opposite impulse on the force sensor with magnitude $J = 2m_B v_{B,2}$. So measuring the impulse allows you to calculate the velocity of the target cart, $v_{B,2} = J/2m_B$. Use your measured values to complete the following table. Velocity entries to the left of the double vertical bar are before the two carts collide; velocity entries to the right are after they collide but before the collision with the force sensor. Hand the table in with your homework solutions.

$m_i [kg]$	$m_t [kg]$	$v_i [m \cdot s^{-1}]$	$v_{cm} [m \cdot s^{-1}]$	$v_i' [m \cdot s^{-1}]$	$J [kg \cdot m \cdot s^{-1}]$	$v_t' [m \cdot s^{-1}]$
0.25	0.25	0.53	0.265	0.00	0.27	0.54
0.25	0.75	0.50	0.125	-0.16	0.31	0.20
0.75	0.25	0.38	0.095	0.17	0.26	0.54

Question 2: Assuming that the collision is elastic, based on the principles of conservation of momentum and conservation of energy, calculate the theoretical final velocities for cart B and cart A for the three trials. Show your work. (These calculations are involved for the second and third trials.) Briefly describe whether or not your measured values agree with your calculated values from part d. Are the collisions elastic?

Elastic collisions: initial $\begin{matrix} m_i \\ \square \end{matrix} \rightarrow v_{i,0} = v_i$ $\begin{matrix} m_t \\ \square \end{matrix} \rightarrow v_{t,0} = 0$ \rightarrow \hat{c}
 final $v_i' \leftarrow \begin{matrix} m_i \\ \square \end{matrix}$ $\begin{matrix} m_t \\ \square \end{matrix} \rightarrow v_t'$

$$b) m_i v_i = m_t v_t' - m_i v_i'$$

$$c) \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i v_i'^2 + \frac{1}{2} m_t v_t'^2$$

i) when $m_i = m_t$

conservation of momentum $\Rightarrow v_i = v_t' - v_i'$ (1)

conservation of energy $\Rightarrow v_i^2 = v_i'^2 + v_t'^2$ (2)

eq (1) $\Rightarrow v_i' = v_t' - v_i$. Substitute into eq (2)

$$v_i^2 = v_t'^2 + (v_t' - v_i)^2 = v_t'^2 + v_t'^2 - 2v_t'v_i + v_i^2$$

Solve for v_t' : $0 = 2v_t'^2 - 2v_t'v_i$

$$\Rightarrow \underbrace{v_t'}_{\text{solution}} = v_i \quad \text{or} \quad v_t' = 0 \quad (\text{initial state})$$

Thus $v_i' = v_t' - v_i = 0$, initial cart stops

ii) when $m_t = 3m_i$

conservation of mom. $\Rightarrow v_i = 3v_t' - v_i'$ (3)

cons. of energy $\Rightarrow v_i^2 = v_i'^2 + 3v_t'^2$ (4)

$$\text{eq (3)} \Rightarrow v_i' = 3v_t' - v_i$$

$$\text{eq (4)} \Rightarrow v_i^2 = (3v_t' - v_i)^2 + 3v_t'^2$$

$$v_i^2 = 9v_t'^2 - 6v_t'v_i + v_i^2 + 3v_t'^2$$

$$\text{or } 0 = 12v_t'^2 - 6v_t'v_i$$

Solution: $\underbrace{v_t' = \frac{1}{2}v_i}_{\text{final state}}, \quad \underbrace{v_t' = 0}_{\text{initial state}}$

$$v_i' = 3v_t' - v_i = \frac{3}{2}v_i - v_i = \frac{1}{2}v_i$$

They both rebound with the same speed but opposite directions.

iii) when $3m_t = m_i$

$$\begin{aligned} \text{cons. of mom} &\Rightarrow 3v_i = v_t' - 3v_i' \quad (5) \\ &\Rightarrow v_t' = 3(v_i + v_i') \end{aligned}$$

$$\text{cons. of energy} \Rightarrow 3v_i^2 = 3v_i'^2 + v_t'^2 \quad (6)$$

which becomes $3v_i^2 = 3v_i'^2 + 9(v_i + v_i')^2$

or

$$0 = 12v_i'^2 + 18v_iv_i' + 6v_i^2$$

this factors $0 = (6)(2v_i' + v_i)(v_i' + v_i)$

Solutions: $\underbrace{v_i' = -\frac{1}{2}v_i}_{\text{final state}}$ or $\underbrace{v_i' = -v_i}_{\text{initial state}}$

initial cart moves in + direction with speed $\frac{1}{2}v_i$

Then $v_t' = 3(v_i + v_i') = 3(v_i - \frac{1}{2}v_i) = \frac{3}{2}v_i$

For the first case, the final velocity of the incident cart was zero, and the target cart was traveling at nearly the initial velocity of the incident cart. The collision was elastic. For the second case, the final velocity of the both the incident cart and the target cart were smaller than the predicted $0.5v_i$ indicating that the collision was not elastic. For the third case, the target cart was 5% slower than the predicted value, while the incident cart was 11% slower, again indicating that the collision was not quite elastic.

Part Three: Modeling the Elastic Collision

Question 3: The repulsive force between the two carts is modeled by the force law

$$F = Ae^{-x/l}$$

where x is the separation between the carts and l is a characteristic length associated with the collision. If the carts start very far apart (an infinite distance), the work required to bring them to a separation distance d is

$$W = \int_{\infty}^d F dx = \int_{\infty}^d Ae^{-x/l} dx.$$

Calculate this integral. The result is the equivalent of the potential energy stored in a spring $(1/2)kx^2$ that satisfies a linear restoring force law.

Solution: The change in potential energy in bringing the carts from an infinite distance apart to a distance d apart is the negative of the work done by the repulsive magnetic force

$$\Delta U = -\int_{\infty}^d F dx = -\int_{\infty}^d Ae^{-x/l} dx = -\frac{Ae^{-x/l}}{(-1/l)} \Big|_{x=\infty}^{x=d} = lAe^{-d/l}.$$

Question 4: Suppose there are two of these carts on a track, one cart with mass m and speed $v_{A,1}$ runs into the other, which has equal mass and is initially at rest. Find the minimum separation d between the carts during the collision. You may assume that A is sufficiently large that so that $d > l$. Evaluate your result numerically for $A = 2.0 \times 10^1 \text{ N}$, $l = 5.0 \times 10^{-3} \text{ m}$, $m = 2.5 \times 10^{-1} \text{ kg}$, and $v_{A,1} = 5.0 \times 10^{-1} \text{ m} \cdot \text{s}^{-1}$. Hint: This problem is most easily done in the center of mass reference frame.

Solution: In the center of mass reference frame, the energy is constant so

$$\Delta K + \Delta U = 0.$$

We already so that the initial kinetic energy in the center of mass frame is given by

$$K_{cm,i} = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} v_i^2 = \frac{1}{4} m v_{A,1}^2.$$

The carts are at rest in the center of mass frame when they reach closet approach.

$$K_{cm,f} = 0.$$

So

$$\Delta K = -\frac{1}{4} m v_{A,1}^2,$$

Thus the constant energy condition becomes

$$-\frac{1}{4} m v_{A,1}^2 + l A e^{-d/l} = 0.$$

Thus

$$e^{-d/l} = \frac{1}{4lA} m v_{A,1}^2.$$

Taking natural ln of both sides and solving for d yields

$$\begin{aligned} d &= -l \ln\left(\frac{1}{4lA} m v_{A,1}^2\right) \\ &= (-5.0 \times 10^{-3} \text{ m}) \ln\left(\frac{(2.5 \times 10^{-1} \text{ kg})(5.0 \times 10^{-1} \text{ m} \cdot \text{s}^{-1})^2}{(4)(5.0 \times 10^{-3} \text{ m})(2.0 \times 10^1 \text{ N})}\right) \\ &= 9.3 \times 10^{-3} \text{ m} \end{aligned}$$