

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_{P_1}^{P_2} (F_x dx + F_y dy + F_z dz)$$

$$\int_{P_1}^{P_2} F_x dx = \int_{P_1}^{P_2} m v_x \frac{dv_x}{dx} dx = \frac{1}{2} m v_x^2 \Big|_{v_{x1}}^{v_{x2}}$$

$$= \frac{1}{2} m (v_{x1}^2 - v_{x2}^2)$$

v_{x1} = velocity-x at P_1

v_{x2} = velocity-x at P_2

$$F_x = m a_x = \frac{dv_x}{dt}$$

$$\int_{P_1}^{P_2} F_x dx = \int_{P_1}^{P_2} m \frac{dv_x}{dt} dx$$

$$\frac{dx}{dt} = \frac{dv_x}{dx} \cdot \frac{dx}{dt} = \frac{dv_x}{dx} \cdot v_x$$

Repeat same for y and z:

$$W = \frac{1}{2} m [v_{x2}^2 + v_{y2}^2 + v_{z2}^2 - (v_{x1}^2 + v_{y1}^2 + v_{z1}^2)]$$

$$W = \frac{1}{2} m (v_2^2 - v_1^2) = K_2 - K_1 = \Delta K \Rightarrow \text{Work-Energy Thm}$$

KE: Potential for particle to do work by virtue of velocity!
W: Work done on particle by the net force equals the change in KE of the particle.

Work
 $W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$



$$s = \frac{1}{2} (v_i + v_f) t$$

$$a = (v_f - v_i) / t$$

v_i = initial velocity

v_f = final velocity

$$W = m \left(\frac{v_f - v_i}{2} \right) (v_i + v_f) t$$

$$W = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2$$

$$W = K_f - K_i$$

where $K = \frac{1}{2} m v^2$

↑ Kinetic Energy of particle with velocity v .

$W > 0$ KE Increases
 $W < 0$ KE Decreases

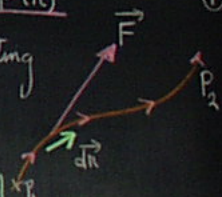
Kinetic Energy: $\vec{F} = \vec{F}(t)$

\vec{F} = Resultant force acting on particle

Particle moves from $P_1 \rightarrow P_2$

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$



$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_{P_1}^{P_2} (F_x dx + F_y dy + F_z dz)$$

$$F_x = ma_x = m \frac{dv_x}{dt}$$

$$\int_{P_1}^{P_2} F_x dx = \int_{P_1}^{P_2} m \frac{dv_x}{dt} dx$$

$$\frac{dx}{dt} = \frac{dv_x}{dx} \cdot \frac{dx}{dt} = \frac{dv_x}{dx} \cdot v_x$$

$$\int_{P_1}^{P_2} F_x dx = \int_{P_1}^{P_2} m v_x \frac{dv_x}{dx} dx = \frac{1}{2} m v_x^2 \Big|_{v_{x1}}^{v_{x2}}$$

$$= \frac{1}{2} m (v_{x1}^2 - v_{x2}^2)$$

v_{x1} = velocity-x at P_1

v_{x2} = velocity-x at P_2

Repeat same for y and z:

$$W = \frac{1}{2} m [v_{x2}^2 + v_{y2}^2 + v_{z2}^2 - (v_{x1}^2 + v_{y1}^2 + v_{z1}^2)]$$

$$W = \frac{1}{2} m (v_2^2 - v_1^2) = K_2 - K_1 = \Delta K \Rightarrow \text{Work-Energy Thm}$$

KE: Potential for particle to do work by virtue of velocity!
W: Work done on particle by the net force equals the change in KE of the particle.

Example:

Move mass down to x_A
Release with upward velocity v_0 .
What is velocity at x_B ?

Net force on block:

$$F = F_T - F_g = mg - kx$$

Work done:

$$W = \int_{x_A}^{x_B} F dx = \int_{x_A}^{x_B} (mg - kx) dx$$

$$W = \left(mgx - \frac{1}{2} kx^2 \right) \Big|_{x_A}^{x_B}$$

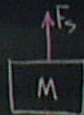
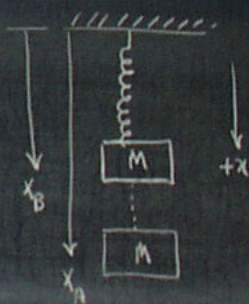
$$= mg(x_B - x_A) - \frac{1}{2} k(x_B^2 - x_A^2)$$

$$= 0.5 \times 9.8(0.2 - 0.5) - \frac{1}{2} 50(0.2^2 - 0.5^2)$$

$$= 3.78 \text{ J}$$

$$W = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2$$

$$\therefore v_B^2 = v_A^2 + \frac{2W}{m} = \left[(-2)^2 + \frac{2 \times 3.78}{0.5} \right] = 14.37 \text{ m/s}^2$$



$m = 0.5 \text{ kg}$
 $k = 50 \text{ N/m}$
 $v_0 = 2 \text{ m/s}$
 $x_A = 0.5 \text{ m}$
 $x_B = 0.2 \text{ m}$

Example:

- Ball in vertical circle.
- Find minimum speed at B so at top, A, tension just vanishes.
- Energy and Dynamics.

At top: $T + mg = \frac{mv_T^2}{R}$

$T = 0$ in limit for min. v .

$$\therefore v_T^2 = gR$$

$W_T = 0$ $T \perp$ displacement.

$$W_g(A \rightarrow B) = mgH = K_B - K_A$$

$$mg(2R) = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_T^2$$

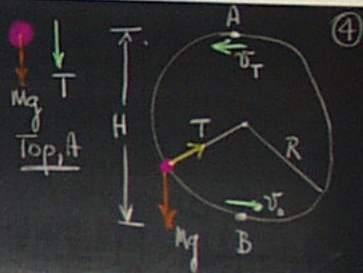
$$v_B^2 = v_T^2 + 4gR$$

$$= gR + 4gR$$

$$= 5gR$$

$$v_B = \sqrt{5gR}$$

If $v < v_B$ will not reach top!
 $> v_B$ $T > 0$



Gravitational Potential Energy

$$W = K_2 - K_1 = \Delta K \quad \text{W-E Theorem}$$

$KE = \frac{1}{2}mv^2$ Capacity to do work due to its velocity.

$PE = U(\vec{r})$ Capacity to do work due to position in space
 Potential Energy!

AND because of position.

$F_g = -mg$ constant.

Move from $(x_1, y_1, z_1) \rightarrow (x_2, y_2, z_2)$

$$W_g = \int_{z_1}^{z_2} \vec{F} \cdot d\vec{z} = \int_{z_1}^{z_2} -mg dz$$

$$= -mgz \Big|_{z_1}^{z_2} = -mg(z_2 - z_1)$$

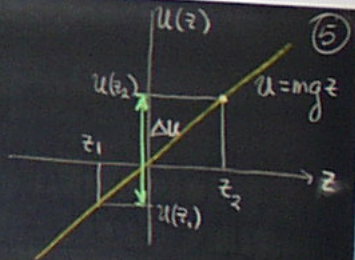
$$= -U(z_2) + U(z_1)$$

where

$$U(z_2) = mgz_2$$

$$U(z_1) = mgz_1$$

$U(z) \equiv$ Gravitational Pot. En.
 = capacity to do work because it is at height z above surface of attracting earth!!



Gravitational Potential Energy

$$W = K_2 - K_1 = \Delta K \quad W-E \text{ Theorem}$$

$KE = \frac{1}{2}mv^2$ Capacity to do work due to its velocity.

$PE = U(\vec{r})$ Capacity to do work due to position in space
Potential Energy!

$$F_g = -mg \text{ constant.}$$

Move from $(x_1, y_1, z_1) \rightarrow (x_2, y_2, z_2)$

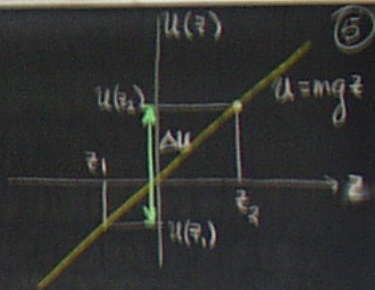
$$\begin{aligned} W_g &= \int_{z_1}^{z_2} \vec{F} \cdot d\vec{z} = \int_{z_1}^{z_2} -mg dz \\ &= -mgz \Big|_{z_1}^{z_2} = -mg(z_2 - z_1) \\ &= -U(z_2) + U(z_1) \end{aligned}$$

where

$$U(z_2) = mgz_2$$

$$U(z_1) = mgz_1$$

$U(z) \equiv$ Gravitational Pot. En.
= capacity to do work
because it is at height
 z above surface of attracting
earth!!



Suppose: Gravity is only force acting.

$$W_g = K_2 - K_1 = -U(z_2) + U(z_1)$$

$$K_2 + U(z_2) = K_1 + U(z_1)$$

$K + U(z) = \text{Constant}$ of the motion

Let $E = K + U(z) = \text{Mechanical Energy}$
= capacity to do work because of velocity
AND because of position.

If only gravity,

$$E = K + U(z) = \text{constant}$$

$$E = \frac{1}{2}mv^2 + mgz = \text{const.}$$

Law of Conservation of Mech. En.

z increases $\Rightarrow v$ decreases

z decreases $\Rightarrow v$ increases

Consider two different positions: z_1 and z_2

$$\frac{1}{2}mv_1^2 + mgz_1 = \frac{1}{2}mv_2^2 + mgz_2$$

$$v_1^2 + 2gz_1 = v_2^2 + 2gz_2$$

$$v_2^2 - v_1^2 = 2g(z_1 - z_2)$$

$$= -2g(z_2 - z_1)$$

Recall constant a kinematics!!

Gravity + Other Forces:

$$W = W_g + W_{\text{other}} = K_2 - K_1 = \Delta K$$

$$W_{\text{other}} - (mgz_2 - mgz_1) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$W_{\text{other}} = \frac{1}{2}(mv_2^2 - mv_1^2) + mg(z_2 - z_1) \\ = \Delta K + \Delta U = \text{change in KE + PE}$$

$$W_{\text{other}} = \left(\frac{1}{2}mv_2^2 + mgz_2\right) - \left(\frac{1}{2}mv_1^2 + mgz_1\right)$$

$$= (K_2 + U_2) - (K_1 + U_1)$$

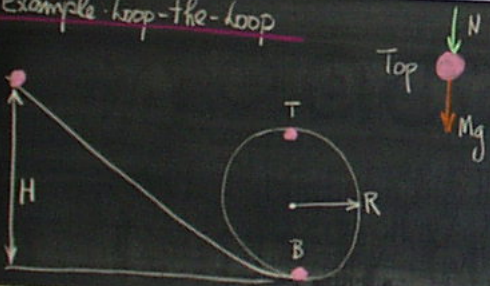
$$= E_2 - E_1 = \Delta E$$

If $W_{\text{other}} > 0$ ME increases.

< 0 ME decreases.

(7)

Example: loop-the-loop



What is H so ball does not fall off?

At top of loop velocity = v_T .

$$F_T = mg + N = \frac{mv_T^2}{R}$$

For min. $v_T \Rightarrow N \equiv 0$.

$$\therefore v_T = \sqrt{gR}$$

Conservation of Energy.

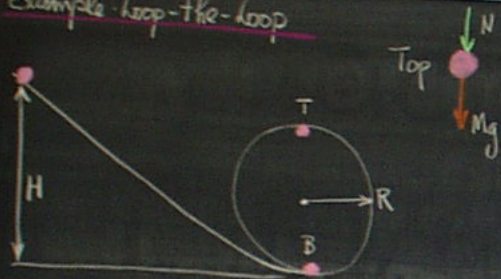
$$mgH + 0 = mg(2R) + \frac{1}{2}mv_T^2$$

$$v_T^2 = 2g(H - 2R)$$

$$v_T = \sqrt{2g(H - 2R)} = \sqrt{gR} \Rightarrow H = \frac{5}{2}R$$

(8)

Example: loop-the-loop



What is H so ball does not fall off?
At top of loop velocity = v_T .

$$F_T = mg + N = \frac{mv_T^2}{R}$$

For min. $v_T \Rightarrow N \equiv 0$.

$$\therefore v_T = \sqrt{gR}$$

Conservation of Energy.

$$mgH + 0 = mg(2R) + \frac{1}{2}mv_T^2$$

$$v_T^2 = 2g(H - 2R)$$

$$v_T = \sqrt{2g(H - 2R)} = \sqrt{gR} \Rightarrow H = \frac{5}{2}R$$

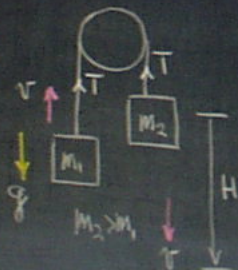
(8)

Atwood's Machine

Release M_2 and let drop
distance H .

$v=0$ at $t=0$

Final $v = ?$



$$W_{m_2} = m_2 g H - T H = \frac{1}{2} m_2 v^2 \quad (1)$$

$$W_{m_1} = -m_1 g H + T H = \frac{1}{2} m_1 v^2 \quad (2)$$

$$(1) + (2) \quad (m_2 - m_1) g H = \frac{1}{2} (m_2 + m_1) v^2$$

$$v = \sqrt{\frac{2(m_2 - m_1)}{m_2 + m_1} g H}$$

Calculate no Forces
No accelerations
No vectors

(9)

Kinetic Energy

14-9

Want to develop a relationship between the work done and the change in speed of a particle.

Particle moves from P_1 to P_2 under the action of a net force \vec{F} (position).

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$$

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$
$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$

$$W = \int_{P_1}^{P_2} (F_x dx + F_y dy + F_z dz)$$

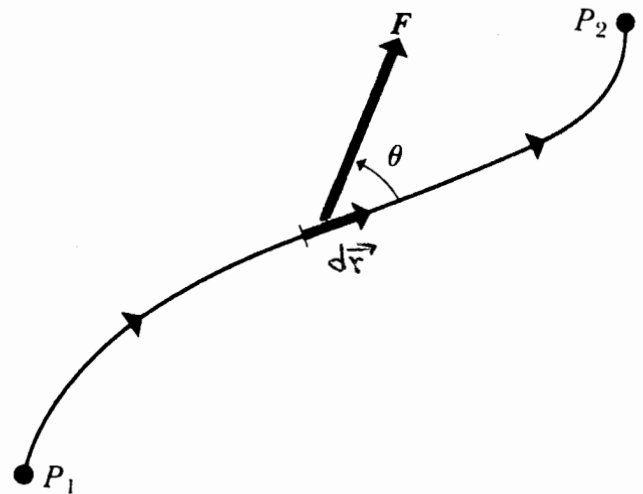
$$F_x = ma_x = m \frac{dv_x}{dt}$$

$$\int_{P_1}^{P_2} F_x dx = \int_{P_1}^{P_2} m \frac{dv_x}{dt} dx$$

↑ v_x is a function of time.

look at v_x as a function of position:

$$\frac{dv_x}{dt} = \frac{dv_x}{dx} \left(\frac{dx}{dt} \right) = \frac{dv_x}{dx} \cdot v_x = v_x \frac{dv_x}{dx}$$



A particle moves along a curved path from point P_1 to P_2 , acted on by a force F that varies in magnitude and direction

$$\begin{aligned}
 \therefore \int_{P_1}^{P_2} F_x dx &= \int_{P_1}^{P_2} m \frac{dv_x}{dt} dx = \int_{P_1}^{P_2} m v_x \frac{dv_x}{dx} dx \\
 &= \int_{P_1}^{P_2} m v_x dv_x = \frac{1}{2} m v_x^2 \Big|_{v_{x1}}^{v_{x2}} \\
 &= \frac{1}{2} m (v_{x2}^2 - v_{x1}^2)
 \end{aligned}$$

v_{x1} = velocity in x-direction at P_1
 v_{x2} = velocity in x-direction at P_2 .

Do the same for terms in y and z.

$$\begin{aligned}
 W &= \frac{1}{2} m \left[v_{x2}^2 + v_{y2}^2 + v_{z2}^2 - (v_{x1}^2 + v_{y1}^2 + v_{z1}^2) \right] \\
 &= \frac{1}{2} m (v_2^2 - v_1^2)
 \end{aligned}$$

$$\boxed{W = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2}$$

Define: $\boxed{K = \frac{1}{2} m v^2}$ \equiv Kinetic Energy of Particle

KE: Potential for a particle to do work by virtue of its velocity.

14-11

The work done on the particle by the net force equals the change in kinetic energy of the particle.

$$W = K_2 - K_1$$

$$\boxed{W = \Delta K} \Rightarrow \text{Work-Energy Theorem.}$$

For a particle $\vec{p} = m\vec{v}$ (Linear Momentum)

$$\therefore K = \frac{1}{2m} p^2$$

Example:

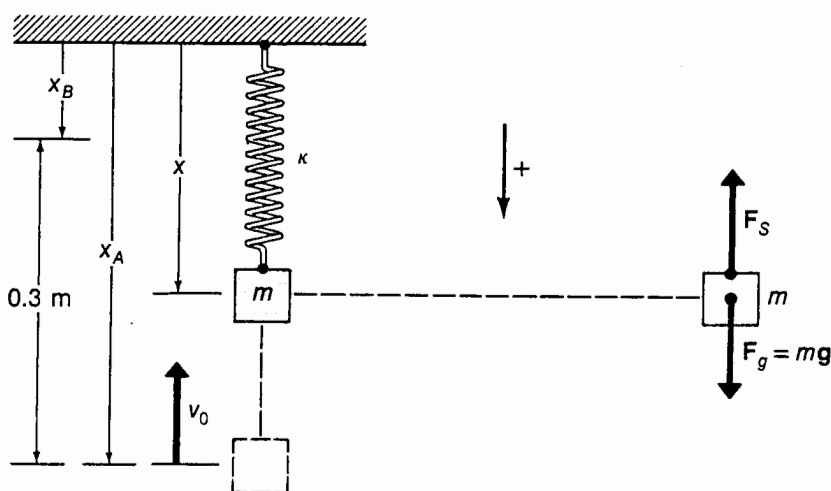
$$m = 0.50 \text{ kg.}$$

$$k = 50 \text{ N/m}$$

$$x_A = 0.50 \text{ m}$$

$$x_B = 0.20 \text{ m}$$

$$v_0 = 2.0 \text{ m/s.}$$



Mass pulled down to x_A , released by imparting an upward velocity v_0 . What is velocity of block at x_B ?

Net force on block:

$$F = F_g - F_s = mg - kx.$$

Work done, integrate $F dx$:

$$W(x_A \rightarrow x_B) = \int_{x_A}^{x_B} F dx = \int_{x_A}^{x_B} (mg - kx) dx = \left(mgx - \frac{1}{2} kx^2 \right) \Big|_{x_A}^{x_B}$$

$$W = mg(x_B - x_A) - \frac{1}{2} k (x_B^2 - x_A^2)$$

$$= 0.5 \times 9.8 (0.2 - 0.5) - \frac{1}{2} \times 50 (0.2^2 - 0.5^2)$$

$$= -1.47 + 5.25 = 3.78 \text{ J}$$

Also $W = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2$

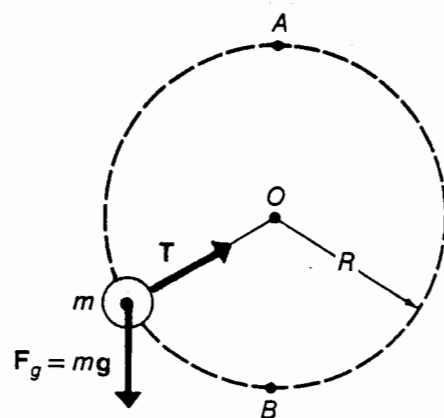
$$\therefore v_B^2 = v_A^2 + \frac{2W}{m} = \left[(-2.0)^2 + \frac{2 \times 3.78}{0.5} \right] \Rightarrow v_B = \pm 4.37 \text{ m/s}$$

Example

14-13

A rock of mass m is tied to the end of a string and whirled in a circular path in the vertical plane.

What is the minimum speed v_0 that the rock has at the bottom (B) if the rock is to pass the top (A) with the string remaining taut?



- Energy considerations alone not sufficient.
- Need to apply Newton's 2nd law.

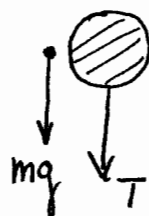
At the top the minimum speed occurs as tension vanishes, $T \rightarrow 0$.

Net downward force is

$$F_g = mg = m \frac{v_T^2}{R}$$

$$v_T^2 = gR$$

[centripetal acceleration in circular path]



$$W(A \rightarrow B) = mgh = K_B - K_A$$

$$mg(2R) = \frac{1}{2}mv_0^2 - \frac{1}{2}mv_T^2$$

$$v_0^2 = v_T^2 + 4gR = gR + 4gR = 5gR$$

$$v_0 = \sqrt{5gR}$$

Example

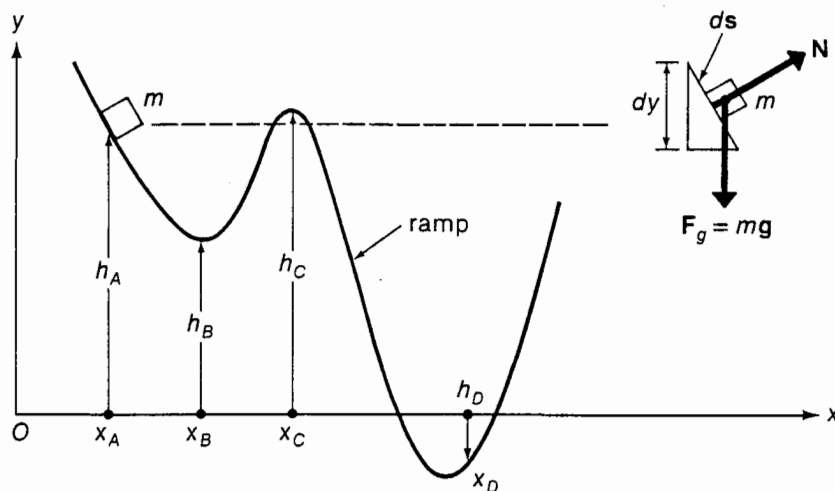
$$h_A = 7\text{m}$$

$$h_B = 4\text{m}$$

$$h_C = 7.2\text{m}$$

$$h_D = -1\text{m}$$

$v_A = 3\text{m/s}$ downward
and tangent to ramp.



What is speed of particle at $x = x_B, x_C, x_D$?

$\vec{N} \cdot d\vec{s} \equiv 0$, so work is done only by gravitational force.

$$W(A \rightarrow B) = \int_A^B \vec{F} \cdot d\vec{s} = \int_A^B (-mg) dy$$

$$W = mg(h_A - h_B)$$

$$\text{Also } W = K_2 - K_1 = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2$$

$$\therefore v_B^2 = v_A^2 + \frac{2W}{m} = v_A^2 + 2g(h_A - h_B)$$

$$= 3^2 + 2 \times 9.8(7 - 4)$$

$$= 67.8 (\text{m/s})^2$$

$$v_B = 8.23 \text{ m/s.}$$

How far up the ramp will particle go after passing $x = x_D$?
 h_m will be reached at x_m where $v_m = 0$.

$$0 = v_A^2 + 2g(h_A - h_m) \Rightarrow h_m = h_A + v_A^2 / 2g \quad h_m = 7.46\text{m}$$

Example

14-15

Car: $m = 1000 \text{ kg}$
 $v_1 = 20 \text{ m/s}$
 $v_2 = 30 \text{ m/s}$

$$W = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

$$= \frac{1}{2} m (v_2^2 - v_1^2)$$

$$= \frac{1}{2} \times 1000 \times (900 - 400)$$

$$= 2.5 \times 10^5 \text{ J}$$

Gravitational Potential Energy

15-1

KE: Represents the capacity of a particle to do work by virtue of its velocity.

PE: Represents the capacity of a particle to do work by virtue of its position in space.

Consider a constant force of gravity

$$\vec{F}_g = -mg \hat{z},$$

acting on a particle which undergoes a displacement from (x_1, y_1, z_1) to (x_2, y_2, z_2) .

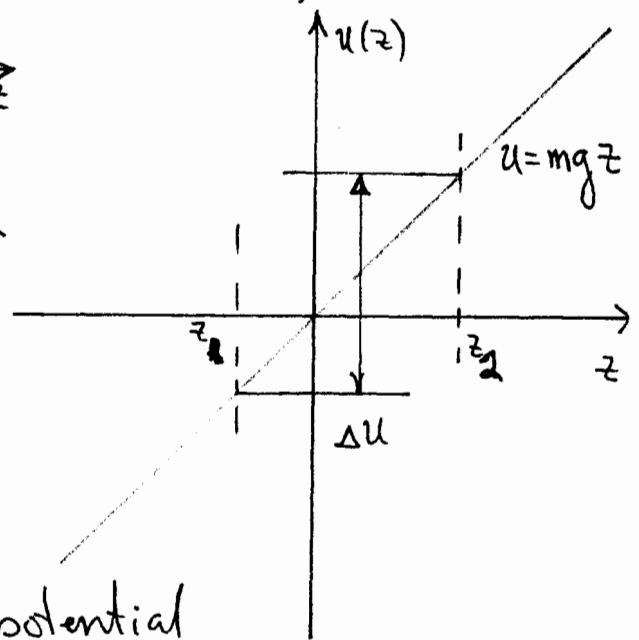
The force does an amount of work;

$$\begin{aligned} W_{gr} &= - \int_{z_1}^{z_2} (z_2 - z_1) = \int_{z_1}^{z_2} \vec{F} \cdot d\vec{z} \\ &= -u(z_2) + u(z_1) = -\Delta u \end{aligned}$$

where,

$$u(z) = mgz,$$

is called the gravitational potential energy.



The change in potential energy between the points z_1 and z_2 is the negative of the work done by gravity on the particle.

Gravitational Potential Energy:

\Rightarrow Capacity of a particle to do work by virtue of its height above the surface of an attracting mass (earth).

If the only force acting is gravity, then using the work-energy theorem,

$$W_{gr} = K_2 - K_1$$

and

$$W_{gr} = -U(z_2) + U(z_1)$$

$$\therefore K_1 + U(z_1) = K_2 + U(z_2)$$

$$\therefore K + U(z) \equiv \text{constant of the motion.}$$

$E = K + U(z)$: Mechanical Energy.

\rightarrow Represents the total capacity of a particle to do work by virtue of both its velocity and its position.

If only force acting is gravity;

$$E = K + U(z) = \text{constant}$$

Law of Conservation of Mechanical Energy

$$E = \frac{1}{2}mv^2 + mgz = \text{constant}$$

z - increases \rightarrow v - decreases

z - decreases \rightarrow v - increases

Consider two different positions:

$$\frac{1}{2}mv_1^2 + mgz_1 = \frac{1}{2}mv_2^2 + mgz_2$$

$$v_1^2 + 2gz_1 = v_2^2 + 2gz_2$$

$$v_2^2 - v_1^2 = 2g(z_1 - z_2)$$

$$= -2g(z_2 - z_1)$$

Recall constant acceleration kinematics results.

Gravity + other forces

15-4

Suppose other forces act besides gravity: friction, etc.
Let W_{other} represent work by all forces ~~other~~
than gravity.

Total work by all forces equals change in KE.

$$W = W_{\text{grav}} + W_{\text{other}} = K_2 - K_1 = \Delta K$$

$$W_{\text{other}} = (mgz_2 - mgz_1) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$W_{\text{other}} = \left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \right) + (mgz_2 - mgz_1)$$

$$= \Delta K + \Delta U$$

↑ change in KE.
↑ change in PE

Also

$$W_{\text{other}} = \left(\frac{1}{2}mv_2^2 + mgz_2 \right) - \left(\frac{1}{2}mv_1^2 + mgz_1 \right)$$

$$= (K_2 + U_2) - (K_1 + U_1)$$

$$= E_2 - E_1 = \Delta E$$

The work done by all other forces acting on the body, with the exception of the gravitational force, equals the change in the total mechanical energy of the body.

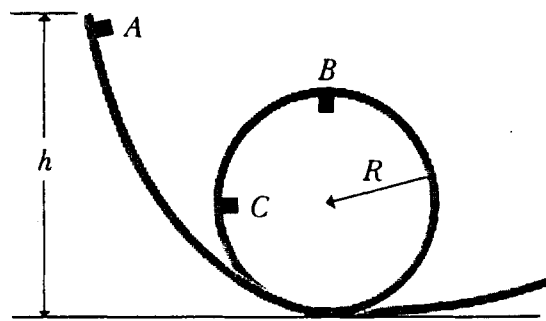
$W_{\text{other}} > 0 \Rightarrow$ Mechanical Energy increases

$W_{\text{other}} < 0 \Rightarrow$ decreases

Example: Loop-the-Loop

15-5

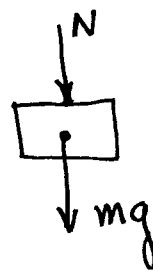
What is the minimum height H (in terms of R) such that an object moves around the loop without falling off the top (B)?



Top of Loop:

velocity: v_T

$$F_c = mg + N = \frac{mv_T^2}{R}$$



Minimum $v_T \Rightarrow N \equiv 0$. [contact force vanishes]

$$\therefore v_T = \sqrt{gR}$$

Represents minimum velocity to make the loop.

Conservation of Energy

$$mgH + 0 = mg(2R) + \frac{1}{2}mv_T^2$$

$$v_T^2 = 2g[H - 2R]$$

$$v_T = \sqrt{2g(H - 2R)}$$

[Energy Considerations]

$$\therefore gR = 2g(H - 2R) \Rightarrow H = \frac{5}{2}R$$