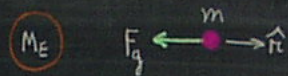


PE: Gravitational (General)

$$\vec{F}_g = -\frac{GmM_E}{r^2} \hat{r} \quad \text{Earth-Object}$$

$$U(P) = -\int_{P_0}^P \vec{F} \cdot d\vec{r} + U(P_0)$$

$$U(r) = \int_{\infty}^r \frac{GmM_E}{r'^2} dr' + U(P_0)$$



$$U(r) = -\frac{GmM_E}{r} \Big|_{\infty}^r + U(P_0)$$

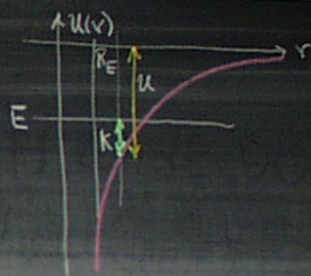
$$\text{Let } U(P_0) = U(\infty) = 0$$

$$U(r) = -\frac{GmM_E}{r}$$

$$E = K + U$$

$$= \frac{1}{2}mv^2 - \frac{GmM_E}{r}$$

Total ME is conserved!



(2)

PE: Spring Force

$$U(P) = -\int_{P_0}^P \vec{F} \cdot d\vec{r} + U(P_0)$$

$$U(x=0) = 0$$

$$U(x) = -\int_0^x F_x(x') dx'$$

$$= -\int_0^x (-kx') dx'$$

$$U(x) = \frac{1}{2}kx^2$$

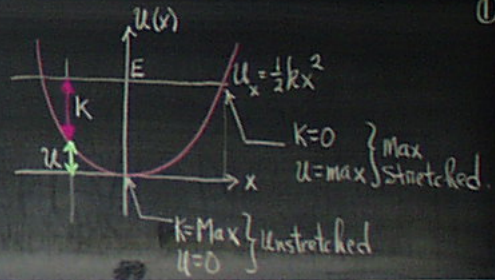
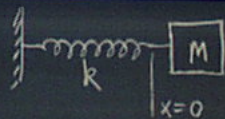
$$E = K + U$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$= \text{constant}$$

Conservative Force

Total ME is conserved!



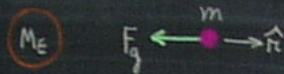
(1)

PE Gravitational (General)

$$\vec{F}_g = -\frac{GmM_E}{r^2} \hat{r} \quad \text{Earth-Object}$$

$$U(r) = -\int_{P_0}^r \vec{F} \cdot d\vec{r} + U(P_0)$$

$$U(r) = \int_{\infty}^r \frac{GmM_E}{r'^2} dr' + U(P_0)$$



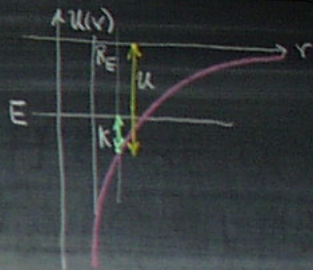
$$U(r) = -\frac{GmM_E}{r} \Big|_{\infty}^r + U(P_0)$$

$$\text{Let } U(P_0) = U(\infty) = 0$$

$$U(r) = -\frac{GmM_E}{r}$$

$$E = K + U \\ = \frac{1}{2}mv^2 - \frac{GmM_E}{r}$$

Total ME is conserved!



(2)

Forces and Potential Energy

Force	$F(x)$	$U(x)$	x_0	$F(x_0)$
Gravity (near)	$mg \hat{j}$	$mg y$	$y=0$	$-mg \hat{j}$
Gravity (far)	$-\frac{GmM_E}{r^2} \hat{r}$	$-\frac{GmM_E}{r}$	$r=\infty$	0
Spring	$-kx \hat{i}$	$\frac{1}{2}kx^2$	$x=0$	0

Non-Cons Forces

$$E_2 - E_1 = W_{\text{Friction}}$$

$$\Delta K + \Delta U = W_{\text{Friction}}$$

$$E_1 = K_1 + U_1$$

$$E_2 = K_2 + U_2$$

$$\left(\frac{1}{2}mv_2^2 + U_2\right) - \left(\frac{1}{2}mv_1^2 + U_1\right) = \int_{x_1}^{x_2} \vec{F} \cdot d\vec{x}$$

Superposition

Several Forces: $\vec{F}_1, \vec{F}_2, \vec{F}_3, \dots$

$$W_{\text{Total}} = \int \vec{F}_1 \cdot d\vec{r} + \int \vec{F}_2 \cdot d\vec{r} + \int \vec{F}_3 \cdot d\vec{r} + \dots$$

$$W_{\text{Total}} = \int \vec{F}_R \cdot d\vec{r} \quad \vec{F}_R \equiv \text{Resultant Force}$$

$$U_{\text{Total}} = U_1 + U_2 + U_3 + \dots$$

$$\therefore K_i + \sum U_i = K_f + \sum U_f \quad \text{Total ME Conserved!}$$

(3)

Example: Simple Pendulum

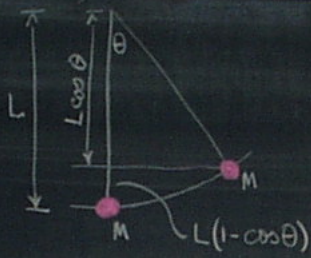
$$U(\theta) = mgL(1 - \cos \theta)$$

$$E = K + U$$

$$= \frac{1}{2}mv^2 + mgL(1 - \cos \theta)$$

For max angle $\theta = \theta_0 \Rightarrow v = 0$

$$\therefore E_0 = mgL(1 - \cos \theta_0)$$



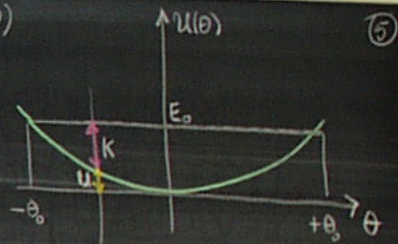
$$\therefore mgL(1 - \cos \theta_0) = \frac{1}{2}mv^2 + mgL(1 - \cos \theta)$$

$$v^2 = 2gL(\cos \theta - \cos \theta_0)$$

At bottom $\theta = 0$

$$v_B = \sqrt{2gL(1 - \cos \theta_0)}$$

At $\theta = \theta_0$ $v \equiv 0$



Example

Drop block M from height H.

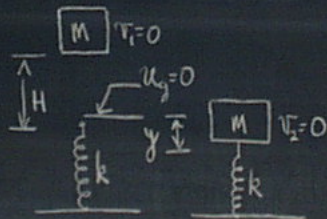
Q: What is max. compression of spring?

All forces cons. (gravity + spring)

$E = K + U = \text{conserved}$

Pos 1: $v_1 = 0$; $K_1 = 0$

Pos 2: $v_2 = 0$; $K_2 = 0$



$$K_1 + U_1 = K_2 + U_2$$

$$0 + mgH = 0 - mgy + \frac{1}{2}ky^2$$

$$y = \frac{1}{2} \left[\frac{2mg}{k} \pm \left(\frac{2mg}{k} \right)^2 + \frac{8mgh}{k} \right]$$

???

Example: Simple Pendulum

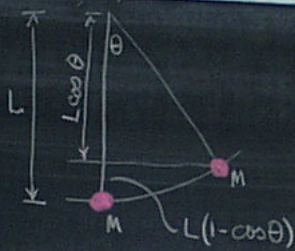
$$U(\theta) = mgL(1 - \cos \theta)$$

$$E = K + U$$

$$= \frac{1}{2}mv^2 + mgL(1 - \cos \theta)$$

For max angle $\theta = \theta_0 \Rightarrow v = 0$

$$\therefore E_0 = mgL(1 - \cos \theta_0)$$



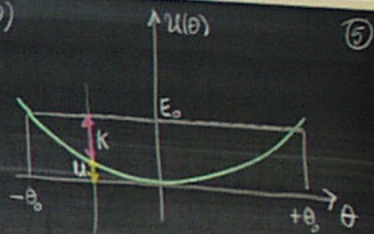
$$mgL(1 - \cos \theta_0) = \frac{1}{2}mv^2 + mgL(1 - \cos \theta)$$

$$v^2 = 2gL(\cos \theta - \cos \theta_0)$$

At bottom $\theta = 0$

$$v_B = \sqrt{2gL(1 - \cos \theta_0)}$$

At $\theta = \theta_0$ $v = 0$



Example: Block + Spring + Friction

Mass released at x_A with $v_A = 0$
Spring is stretched.

Mass moves to left.

What is velocity v_B at $x = x_B$?

Force of friction $f = \mu_k N = \mu_k mg$

Work done by 'f' $W_f = \int_{x_A}^{x_B} \vec{f} \cdot d\vec{x} = \mu_k mg(x_B - x_A)$



Energy Conserved with non-conservative forces.

$$\frac{1}{2}mv_B^2 + \frac{1}{2}kx_B^2 - \left[\frac{1}{2}mv_A^2 + \frac{1}{2}kx_A^2 \right] = W_f = \mu_k mg(x_B - x_A)$$

$$\frac{1}{2}mv_B^2 = \underbrace{\frac{1}{2}k(x_A^2 - x_B^2)}_{>0} + \underbrace{\mu_k mg(x_B - x_A)}_{<0}$$

$x_A > x_B$

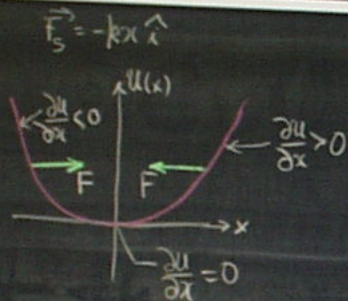
Example: Spring Force

$$U(x) = \frac{1}{2} kx^2$$

$$F_x = -\frac{\partial U}{\partial x} = -kx$$

$$F_y = -\frac{\partial U}{\partial y} = 0$$

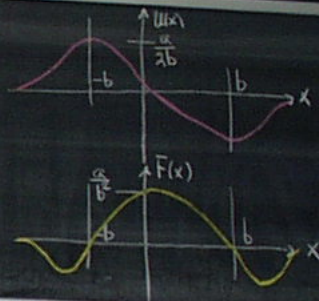
$$F_z = -\frac{\partial U}{\partial z} = 0$$



Example

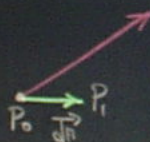
$$U(x) = \frac{-ax}{b^2 + x^2}$$

$$F(x) = -\frac{\partial U}{\partial x} = \frac{a(b^2 - x^2)}{(b^2 + x^2)^2}$$



Force $\longleftrightarrow$ Potential Energy

$$U(P_1) - U(P_0) = \int_{P_0}^{P_1} \vec{F} \cdot d\vec{r}$$



Potential $E \rightarrow$ Force?

$$\begin{aligned} dU &= U(P_1) - U(P_0) = -\vec{F} \cdot d\vec{r} \\ &= -F_x dx - F_y dy - F_z dz \end{aligned}$$

Assume $dy=0, dz=0$

$$\text{Then } dU = -F_x dx$$

$$\therefore F_x = -\frac{dU}{dx} \quad \left\{ \begin{array}{l} \text{Diff. } U(x,y,z) \\ \text{Keep } y, z = \text{constant.} \end{array} \right.$$

Define: **Partial Derivatives**

$$F_x = -\frac{\partial U}{\partial x} \quad F_y = -\frac{\partial U}{\partial y} \quad F_z = -\frac{\partial U}{\partial z}$$

$$\vec{F} = \left[\frac{\partial U}{\partial x} \hat{i} + \frac{\partial U}{\partial y} \hat{j} + \frac{\partial U}{\partial z} \hat{k} \right] = -\vec{\nabla} U(x,y,z) \quad \vec{\nabla} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right)$$

Gradient Vector Operator

Example: Spring Force

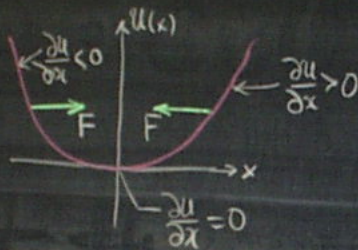
$$U(x) = \frac{1}{2} k x^2$$

$$F_x = -\frac{\partial U}{\partial x} = -kx$$

$$F_y = -\frac{\partial U}{\partial y} = 0$$

$$F_z = -\frac{\partial U}{\partial z} = 0$$

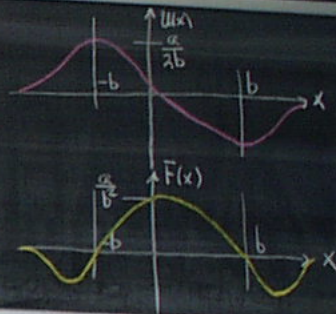
$$\vec{F}_s = -kx \hat{x}$$



Example

$$U(x) = \frac{-ax}{b^2 + x^2}$$

$$F(x) = -\frac{\partial U}{\partial x} = \frac{a(b^2 - x^2)}{(b^2 + x^2)^2}$$



Example

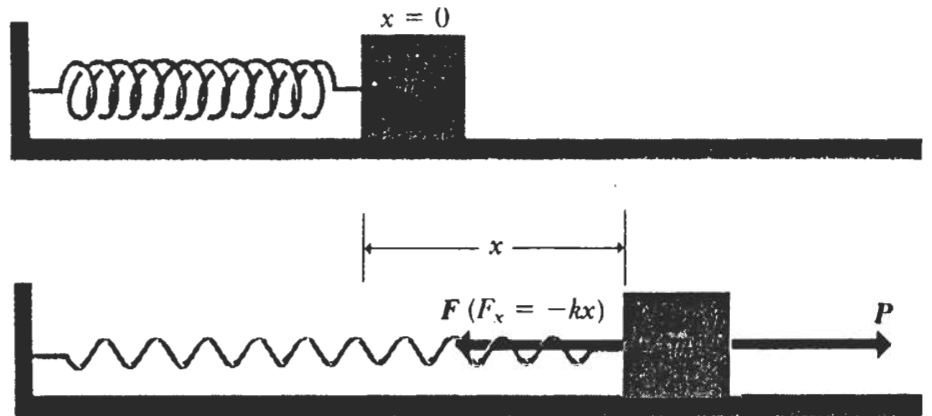
$$U(x, y) = Ax^2y^2$$

$$F_x = -\frac{\partial U}{\partial x} = -2Ax^2y^2$$

$$F_y = -\frac{\partial U}{\partial y} = -2Ax^2y$$

PE: Spring

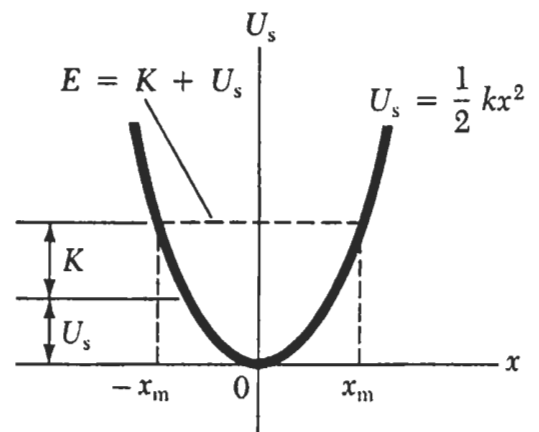
16-7



For reference point we choose point P_0 at $x=0$, spring is in equilibrium.

$$u(P_0) \equiv 0.$$

$$u(x) = - \int_0^x F_x(x') dx' \\ = - \int_0^x (-kx') dx'$$



$$u(x) = \frac{1}{2} kx^2$$

The total mechanical energy for the mass-spring system is :

$$E = K + u = \frac{1}{2} mv^2 + \frac{1}{2} kx^2$$

PE: Gravitational (General)

16-8

$$\vec{F}_g = -\frac{G m M_E}{r^2} \hat{r}$$

$$U(P) = -\int_{P_0}^P \vec{F} \cdot d\vec{r} + U(P_0)$$

$$U(r) = \int_P^r \frac{G m M_E}{r'^2} dr' + U(P_0)$$

$$U(r) = -\frac{G m M_E}{r} \Big|_{\infty}^r + U(P_0)$$

$$U(P_0) = U(\infty) = 0$$

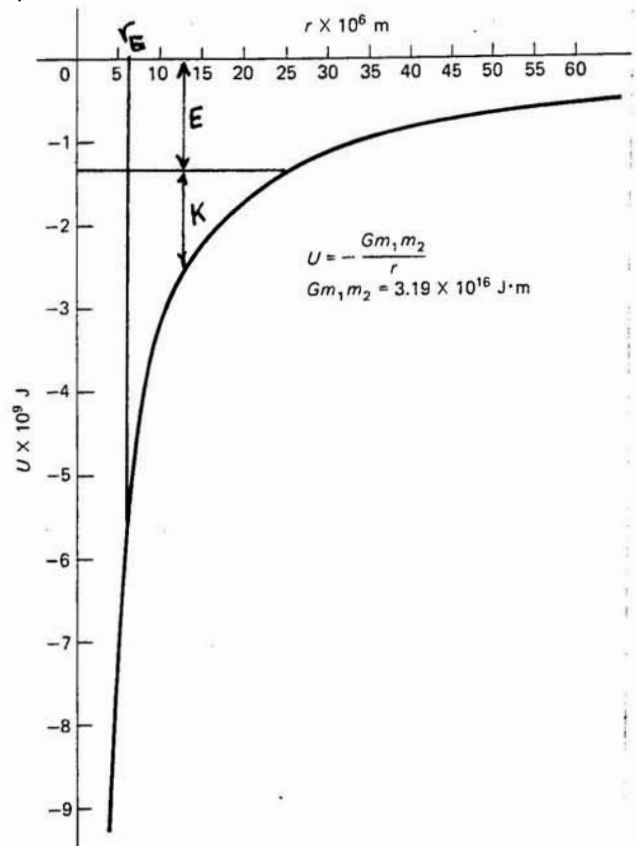
$$U(r) = -\frac{G m M_E}{r}$$

Total Mechanical Energy

$$E = K + U = \frac{1}{2} m v^2 - \frac{G m M_E}{r}$$



The mutual gravitational potential energy $U = -Gm_1m_2/r$ between an 80-kg object and the earth is shown versus their separation r .



Forces $\longleftrightarrow$ Potential Energy

<u>Force</u>	$\vec{F}(x)$	$U(x)$	x_0	$F(x_0)$
gravity (near)	$-mg \hat{j}$	mgy	$y=0$	$-mg \hat{j}$
spring	$-kx \hat{i}$	$\frac{1}{2} kx^2$	$x=0$	0
gravity (far)	$-\frac{GmM_E}{r^2} \hat{r}$	$-\frac{GmM_E}{r}$	$r=\infty$	0

often convenient to choose reference point where $\vec{F}(x_0) = 0$ and $U(x_0) \equiv 0$.

$$W_{if}(\text{Cons.}) + W_{if}(\text{N-Cons}) = K_f - K_i \quad [\text{WE Theorem}]$$

$$W_{if}(\text{NCons}) = K_f - K_i + (-W_{if}(\text{N-Cons}))$$

$$= K_f - K_i + U_f - U_i$$

$$W_{if}(\text{N-Cons}) = E_f - E_i \quad [\text{modified } W-E]$$

$$E = K + U \quad \text{Total m.E.}$$

$$\underline{E_f = K_f + U_f = E_i = K_i + U_i} \quad \underline{\text{if}} \quad [W_{if}(\text{N-Cons}) \equiv 0]$$

Superposition

16-9

Several conservative forces acting on an object:

$$\vec{F}_A, \vec{F}_B, \vec{F}_C$$

$$W_{\text{total}} = \int \vec{F}_A \cdot d\vec{r} + \int \vec{F}_B \cdot d\vec{r} + \int \vec{F}_C \cdot d\vec{r}$$

$$W_{\text{total}} = \int \vec{F}_R \cdot d\vec{r}$$

$\vec{F}_R \equiv$ Resultant Force.

$$U = U_A + U_B + U_C$$

Net Potential Energy $\rightarrow$ sum of all the individual potential energies for each force.

Total mechanical energy is conserved.

$$\boxed{K_i + \sum U_i = K_f + \sum U_f = E}$$

Example

(See notes 14-12)

$$m = 0.50 \text{ kg}$$

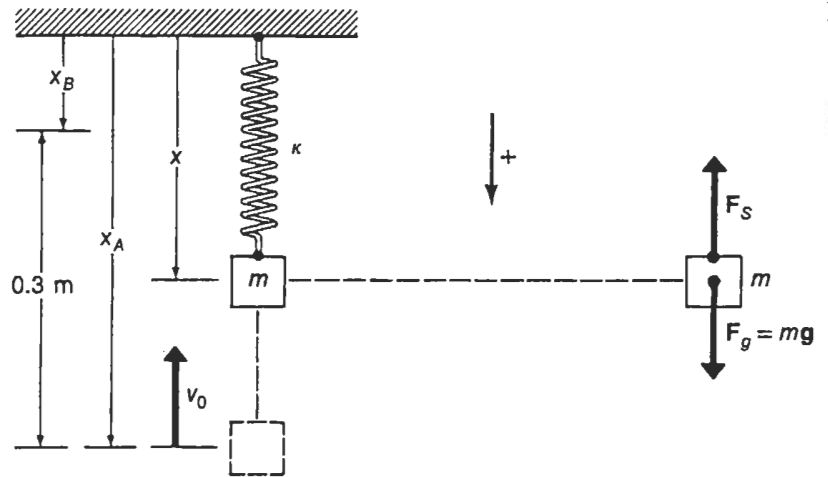
$$K = 50 \text{ N/m}$$

$$x_A = 0.50 \text{ m}$$

$$x_B = 0.20 \text{ m}$$

$$v_0 = 2.0 \text{ m/s} \uparrow$$

$$v_B = ?$$



Energy considerations:

2 PE's [gravity + spring]

$$\frac{1}{2} m v_A^2 + u(x_A) = \frac{1}{2} m v_B^2 + u(x_B)$$

$$\underbrace{\frac{1}{2} m v_A^2}_{\text{KE}} - \underbrace{mg x_A + \frac{1}{2} K x_A^2}_{\text{PE}} = \frac{1}{2} m v_B^2 - mg x_B + \frac{1}{2} K x_B^2$$

Solve for v_B :

$$\begin{aligned} v_B^2 &= v_A^2 - 2g(x_A - x_B) + \left(\frac{K}{m}\right)(x_A^2 - x_B^2) \\ &= 2.0^2 - 2 \times 9.8(0.50 - 0.20) + \frac{50}{0.5}(.50^2 - .20^2) \\ &= 19.12 \end{aligned}$$

$$v_B = 4.37 \text{ m/s}$$

Example

$$m = 2.6 \text{ kg}$$

$$k = 72 \text{ N/m}$$

$$h = 0.55 \text{ m}$$

Mass m dropped from rest on spring. What is maximum compression of spring?

All conservative forces, PE represents all work done.

$$E = K + U = \text{conserved}$$

At release, brick is at rest, $v_1 = 0$ and $K_1 = 0$.

At maximum compression, brick is also momentarily at rest, $K_2 = 0$

$$K_1 + U_1 = K_2 + U_2$$

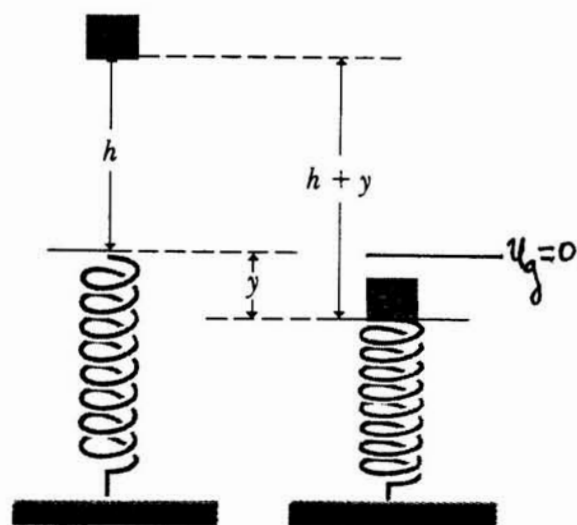
$$0 + mgh = 0 - mg(-y) + \frac{1}{2}ky^2$$

$$y^2 - \frac{2mg}{k}y - \frac{2mgh}{k} = 0$$

$$y = \frac{1}{2} \left[\frac{2mg}{k} \pm \sqrt{\left(\frac{2mg}{k}\right)^2 + \frac{8mgh}{k}} \right]$$

+ $\rightarrow$ Answer.
- $\rightarrow$ [Mass + spring together str spring other side]

Dropping a brick onto a spring-mounted platform: using elastic and gravitational potential energies together.



Initial position

Maximum compression

The total fall of the block is $h + y$.

Example - Simple Pendulum

16-12

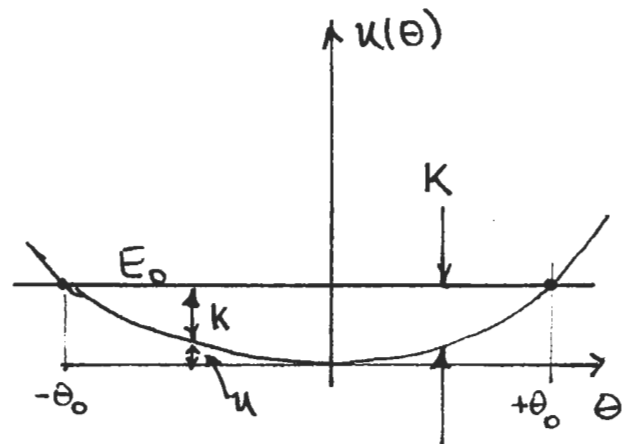
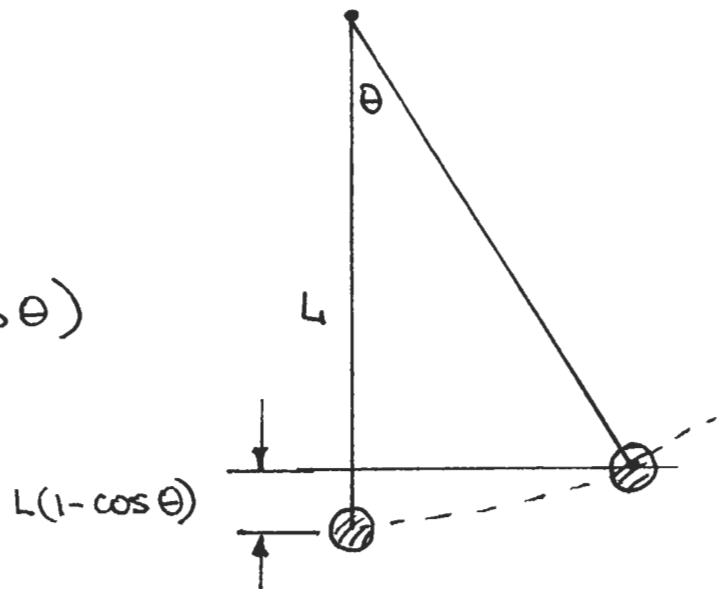
$$u(\theta) = mgL(1 - \cos \theta)$$

$$E = K + u$$

$$= \frac{1}{2}mv^2 + mgL(1 - \cos \theta)$$

For max. angle
 $\theta = \theta_0$ $v = 0$.

$$\therefore E_0 = mgL(1 - \cos \theta_0)$$



$$\therefore mgL(1 - \cos \theta_0) = \frac{1}{2}mv^2 + mgL(1 - \cos \theta)$$

$$v^2 = 2gL(\cos \theta - \cos \theta_0)$$

At the bottom, $\theta = 0$

$$v_B = \sqrt{2gL(1 - \cos \theta_0)}$$

Non-Conservative Forces

16-13

If non-conservative forces act on an object, then the change in the KE + PE of the conservative force will be equal to the work done by the friction force.

$$\Delta K + \Delta U = W_{\text{Friction}}$$

$\Delta K \equiv$ change in KE

$\Delta U \equiv$ change in PE

$W_f \equiv$ work done by frictional force.

$$E_1 = K_1 + U_1$$

$$E_2 = K_2 + U_2$$

$$(E_2 - E_1) = W_{\text{Friction}}$$

$$\frac{1}{2}mv_2^2 + U(x_2) - \left[\frac{1}{2}mv_1^2 + U(x_1) \right] = \int_{x_1}^{x_2} f dx$$

Example

16-14

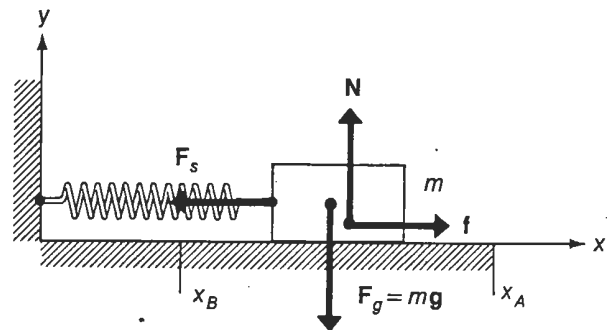
$$m = 0.50 \text{ kg}$$

$$k = 50 \text{ N/m}$$

$$\mu_k = 0.20$$

$$x_A = 0.30 \text{ m}$$

$$x_B = 0.05 \text{ m}$$



Mass released at $x = x_A$ with $v_A = 0$ [Spring is stretched]
What is velocity, v_B when $x = x_B$?

$$\text{Force of friction } f = \mu_k N = \mu_k mg$$

Work done by friction

$$W_f = \int_{x_A}^{x_B} \vec{f} \cdot d\vec{x} = \overbrace{+\mu_k mg (x_B - x_A)}^{-0.245 \text{ J}}$$

Energy Conservation:

$$\left[\frac{1}{2} m v_B^2 + \frac{1}{2} k x_B^2 \right] - \left[\cancel{\frac{1}{2} m v_A^2} + \frac{1}{2} k x_A^2 \right] = \mu_k mg (x_B - x_A)$$

$$\frac{1}{2} m v_B^2 = \frac{1}{2} k (x_A^2 - x_B^2) + \mu_k mg (x_B - x_A)$$

$$= \frac{1}{2} \times 50 (.3^2 - .05^2) - 0.245$$

$$v_B = 2.788 \text{ m/s}$$

Force $\longleftrightarrow$ Potential Energy

16-15

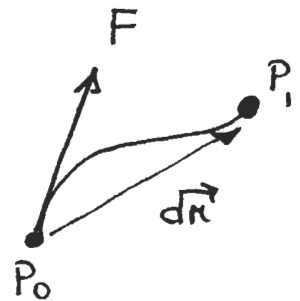
We have seen how to calculate the PE given a conservative force:

$$u(P_1) - u(P_0) = - \int_{P_0}^{P_1} \vec{F} \cdot d\vec{r}$$

Can we calculate the force given the PE?

Yes !!!

Assume P_0 and P_1 are separated by the infinitesimal displacement $d\vec{r}$, then differentiating expression for $u(P_1)$:



$$\begin{aligned} du &= u(P_1) - u(P_0) = - \vec{F} \cdot d\vec{r} \quad [\text{Inverse of P.E.}] \\ &= -F_x dx - F_y dy - F_z dz \end{aligned}$$

Assume displacement is only along x ,
 $dy=0$, $dz=0$.

$$du = -F_x dx$$

$$\text{or } F_x = - \frac{du}{dx} \left. \vphantom{\frac{du}{dx}} \right\} \text{differentiate keeping } y, z \text{ constant.}$$

Define a special derivative called a partial derivative for one variable at a time,

$$\left. \begin{aligned} F_x &= -\frac{\partial U}{\partial x} \\ F_y &= -\frac{\partial U}{\partial y} \\ F_z &= -\frac{\partial U}{\partial z} \end{aligned} \right\} \text{Combining}$$

$$\vec{F} = -\left(\frac{\partial U}{\partial x} \hat{x} + \frac{\partial U}{\partial y} \hat{y} + \frac{\partial U}{\partial z} \hat{z} \right)$$

$$= -\vec{\nabla} U$$

Example: Spring force

$$U(x) = \frac{1}{2} kx^2$$

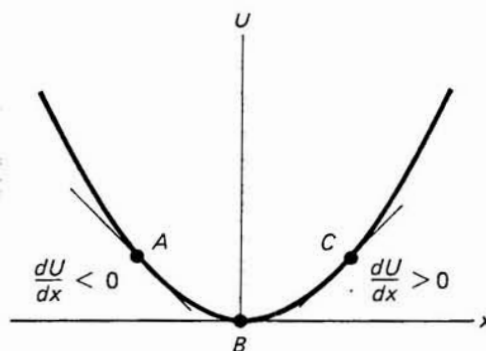
PE of an elastic spring.

$$F_x = -\frac{\partial U}{\partial x} = -kx$$

$$F_y = -\frac{\partial U}{\partial y} = 0$$

$$F_z = -\frac{\partial U}{\partial z} = 0$$

The potential energy $U = \frac{1}{2} kx^2$ is shown for a spring. When the spring is compressed, $x < 0$, the slope is negative, and the force is positive. When the spring is stretched, $x > 0$, the slope is positive, and the force is negative.

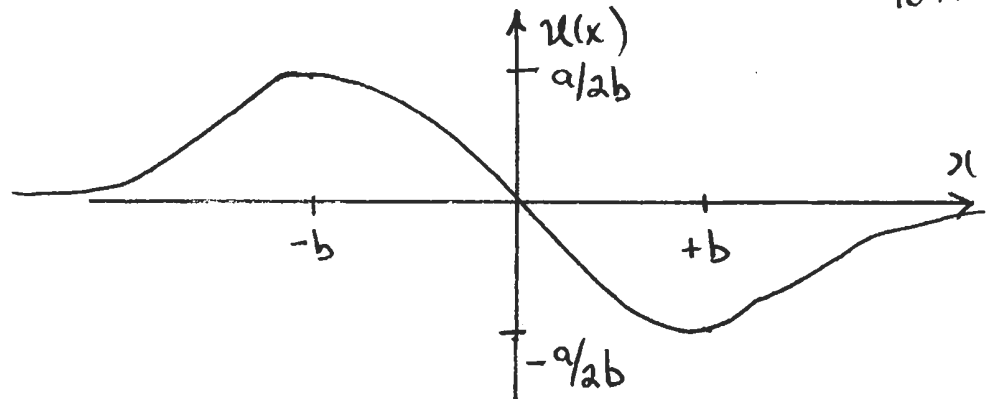


$$\vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \quad [\text{Vector Operator}]$$

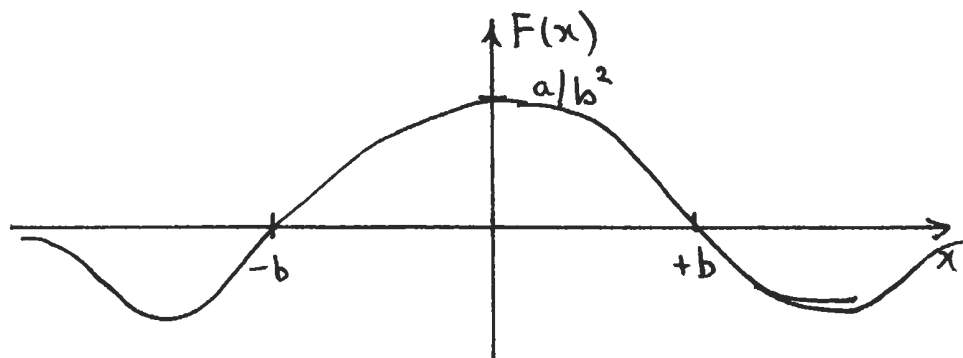
Example

16-17

$$u(x) = \frac{-ax}{b^2 + x^2}$$



$$F(x) = -\frac{\partial u(x)}{\partial x}$$
$$= \frac{a(b^2 - x^2)}{(b^2 + x^2)^2}$$



Example

$$u(x,y) = Ax^2y^2$$

$$F_x = -\frac{\partial u}{\partial x} = -2Axy^2$$

$$F_y = -\frac{\partial u}{\partial y} = -2Ax^2y$$