

## Energy Curves:

Given  $U(x)$

Want  $x(t)$

Assumes conservative forces only

Then  $E = K + U \equiv \text{constant}$

$$= \frac{1}{2}mv^2 + U(x)$$

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U(x)$$

$$v = \frac{dx}{dt} = \sqrt{\frac{2}{m}[E - U(x)]}$$

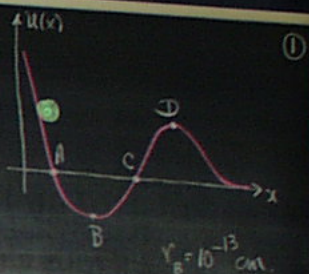
$$\int_{x'=x_0}^x \frac{dx'}{\sqrt{\frac{2}{m}[E - U(x') ]}} = \int_{t=0}^t dt'$$

where  $x' = x_0$  at  $t' = 0$

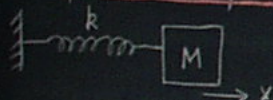
Not easy to solve  
Computationally, numerically!

## Example: Alpha Particle + Au Nucleus

1.  $x_1, x_2: \frac{\partial U}{\partial x} = 0 \quad F(x) = 0$
2.  $\frac{\partial U}{\partial x} \rightarrow 0$  as  $x \rightarrow \infty \quad F(x) \rightarrow 0$
3.  $F(x) > 0 \quad 0 < x < x_2$  Repulsive Force.
4.  $F(x) < 0 \quad x_0 < x < x_1$  Attractive Force.



## Example: Mass on Spring



$$E = K + U = \frac{1}{2}kx_0^2 \quad \text{Initial Total } E$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_0^2$$

$$i) x > 0 \quad \frac{\partial U}{\partial x} = kx > 0 \quad F < 0 \quad \text{Attractive}$$

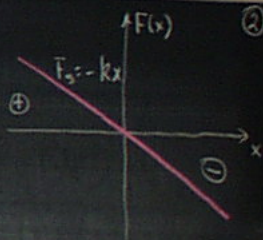
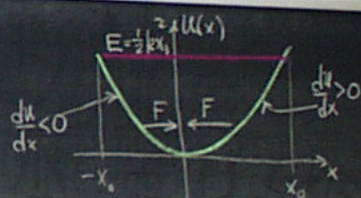
$$ii) x < 0 \quad \frac{\partial U}{\partial x} = kx < 0 \quad F > 0 \quad \text{Attractive}$$

$$iii) x = 0 \quad \frac{\partial U}{\partial x} = 0 \quad F = 0 \quad \text{Neutral}$$

## Harmonic Oscillation!

Motion bounded

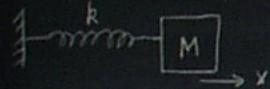
Maxes between turning points determined by total  $E$ .



$$\text{Unbounded } x_0 < x' < \infty$$

$$x \rightarrow \infty \quad \frac{\partial U}{\partial x} = 0 \quad \text{Neutral.}$$

### Example: Mass on Spring



$$E = K + U = \frac{1}{2} k x_0^2 \quad \text{Initial Total } E$$

$$= \frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \frac{1}{2} k x_0^2$$

i)  $x > 0 \quad \frac{\partial U}{\partial x} = kx > 0 \quad F < 0$  Attractive

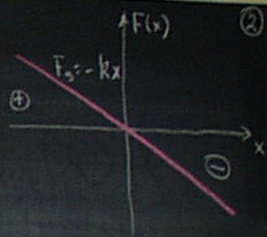
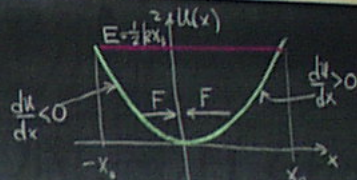
ii)  $x < 0 \quad \frac{\partial U}{\partial x} = -kx < 0 \quad F > 0$  Attractive

iii)  $x = 0 \quad \frac{\partial U}{\partial x} = 0 \quad F = 0$  Neutral

Harmonic Oscillator!

Motion bounded

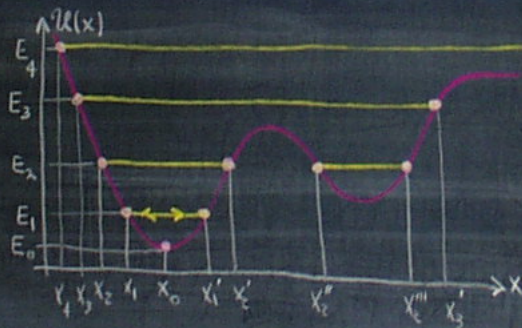
Moves between turning points determined by total  $E$ .



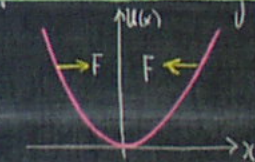
### Example: Potential Well

$$v_x = \sqrt{\frac{2}{m} [E - U(x)]}$$

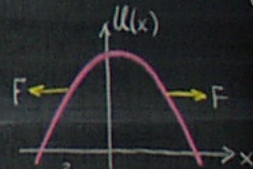
- i)  $E = E_0$  Fixed pos.  $K=0$
  - ii)  $E = E_1$  Oscillates  $x_1 < x < x_1'$
  - iii)  $E = E_2$  4 turning points
  - iv)  $E = E_4$  1 turning point
- Unbounded  $x_4 < x < \infty$



### Equilibrium/Stability



$$\frac{\partial^2 U}{\partial x^2} > 0 \quad \text{Pot. Minimum Stable!}$$



$$\frac{\partial^2 U}{\partial x^2} < 0 \quad \text{Pot. Max Unstable.}$$

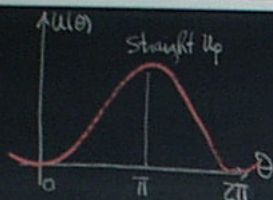
$$\frac{\partial^2 U}{\partial x^2} = 0 \quad \text{Neutral.}$$

# Pendulum



$$U(\theta) = mgl(1 - \cos\theta)$$

$$U(0) = 0 \quad U(\pi) = 2mgl$$



$$\theta = 0 \quad \frac{\partial^2 U}{\partial \theta^2} > 0 \text{ Stable.}$$

$$\theta = \pi \quad \frac{\partial^2 U}{\partial \theta^2} < 0 \text{ Unstable.}$$

## Example Diatomic Molecule

Lennard-Jones Potential (6,12)

$$U(r) = U_0 \left[ \left( \frac{a}{r} \right)^{12} - \left( \frac{a}{r} \right)^6 \right]$$

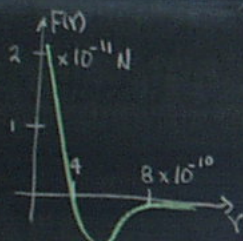
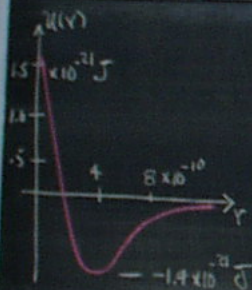
$$F(r) = -\frac{dU}{dr} = \frac{6U_0}{a} \left[ 2 \left( \frac{a}{r} \right)^{13} - \left( \frac{a}{r} \right)^7 \right]$$

$$\text{At equl: } F(r) = 0 \Rightarrow \left( \frac{a}{r_0} \right)^6 = \frac{1}{2} \Rightarrow r_0 = 3.9 \times 10^{-10} \text{ m}$$



$$\left. \begin{aligned} U_0 &= 5.6 \times 10^{-21} \text{ J} \\ a &= 3.5 \times 10^{-10} \text{ m} \end{aligned} \right\} 0$$

(4)



## Example: Repulsive Square Law Force

$$F(x) = \frac{A}{x^2} \quad A > 0$$

$$U(x) = \frac{A}{x}$$

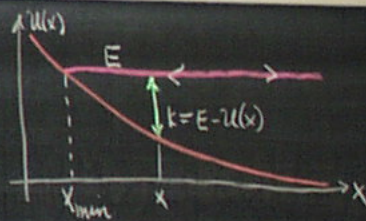
Like two equal charges.

Assume total energy is E

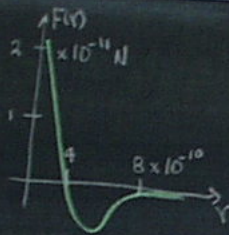
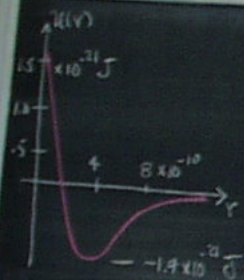
Then minimum x:

$$x_{\min} = \frac{A}{U} = \frac{A}{E}$$

$$\text{i.e. } U(x_{\min}) = E$$



(5)



Example Repulsive Square Law Force

$$F(x) = \frac{A}{x^2} \quad A > 0$$

$$U(x) = \frac{A}{x}$$

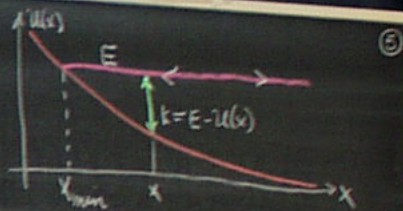
Like two equal charges.

Assume total energy is  $E$

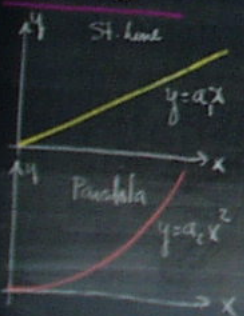
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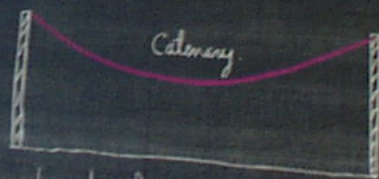
Least Time:



Catenary.

$$y = \frac{1}{2a}(e^{ax} + e^{-ax})$$

$$= \frac{1}{a} \cosh ax$$



- Lowest c.p.g.
- Smallest PE
- Nature slopes curve to minimize PE

Power : Time Rate of Doing Work

$$\bar{P} = \frac{\Delta W}{\Delta t} \quad \text{Average Power}$$

$$P(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} \quad \text{Inst. Power}$$

$$[P] = \text{J/s} = \text{Watts}$$

$$1 \text{ HP} = 550 \text{ ft} \cdot \text{lb/s} = 745.7 \text{ W}$$

Power  $\leftrightarrow$  Force

$$dW = \vec{F} \cdot d\vec{r}$$

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v} \quad [\vec{F} = \text{const.}]$$

Power Depends on Reference Frames.  
Since velocity depends on frame.

Energy  $\leftrightarrow$  Power

$$E = \int_{t_1}^{t_2} P(t) dt = Pt \quad \text{for constant } P$$

$$1 \text{ kWh} = 3.6 \times 10^6 \text{ J} = 1000 \text{ W for 1-hour}$$

$$1 \text{ Kcalorie} = 4.19 \times 10^3 \text{ J (foods)}$$

Example: Car up Incline

Moves at constant  $\vec{v}$

$$F - f + mg \sin \theta = 0 \quad [a=0]$$

$$F = 700 + 1400 \times 9.8 \times \sin 10^\circ$$
$$= 3100 \text{ N}$$

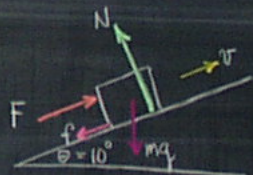
$$P = \vec{F} \cdot \vec{v} \quad [\vec{F} \parallel \vec{v}]$$

$$= Fv$$

$$= 3100 \times 22$$

$$= 6.8 \times 10^4 \text{ W}$$

$$= 91 \text{ HP}$$



$$M = 1400 \text{ kg}$$

$$v = 22 \text{ m/s (80 km/h)}$$

$$f = 700 \text{ N friction}$$

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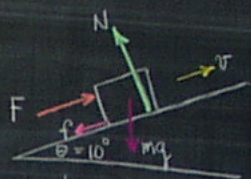
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$$M = 1400 \text{ kg}$$

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### Example: Elevator

$$T - f - Mg = 0$$

$$T = f + Mg = 4 \times 10^3 + 1.8 \times 10^3 \times 9.8$$
$$= 2.16 \times 10^4 \text{ N}$$

$$P = \vec{T} \cdot \vec{v} = T v = 2.16 \times 10^4 \times 3$$
$$= 6.48 \times 10^4 \text{ W} = 86.9 \text{ HP.}$$

Suppose  $a = 1.0 \text{ m/s}^2$

$$T - f - Mg = ma$$

$$T = M(a + g) + f$$

$$= 2.34 \times 10^4 \text{ N}$$

$$P = T v$$

$$= 2.34 \times 10^4 v$$

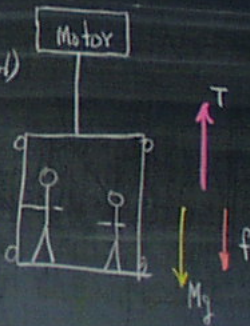
Power increases as speed increases.

$$M(\text{elevator} + \text{load})$$

$$= 1800 \text{ kg.}$$

$$f = 4000 \text{ N}$$

$$v = 3 \text{ m/s} \uparrow$$



## Forces in Equilibrium

27-1

We want to determine the conditions under which rigid bodies are in equilibrium.

Static Equilibrium: Objects are completely at rest.

Dynamic Equilibrium: Objects may have constant linear and/or angular velocity.

- Particles were seen to be in equilibrium when the sum of the forces acting on them is zero.

- Rigid bodies require in addition that the sum of the torques about any axis must be zero for the body not to have a tendency to rotate.

$$\vec{P}_{cm} = \text{constant}$$

$$\vec{L} = \text{constant}$$

i) First condition for equilibrium:

$$\sum \vec{F}_e = \frac{d\vec{P}}{dt} = 0$$

$$\vec{P}_{cm} = \text{constant}$$

The resultant external force acting on the body must be zero.

In components,

$$\sum F_x = 0, \sum F_y = 0, \sum F_z = 0.$$

This is a statement of translational equilibrium, i.e. the linear acceleration of the cm of the body must be zero when viewed from an inertial reference frame.

ii) Second condition for equilibrium:

$$\sum \vec{\tau}_e = \frac{d\vec{L}}{dt} = 0 \quad \vec{L} = \text{constant}$$

$$\Rightarrow \vec{\alpha} = 0$$

The sum of all the external torques acting on the body about any axis must be zero.

In components,  $\sum \tau_x = 0$ ,  $\sum \tau_y = 0$ , and  $\sum \tau_z = 0$ .

This is a statement of rotational equilibrium, i.e. the angular acceleration about any axis must be zero.

General Case:

- 3 Force Equations
- 3 Torque Equations.

Special Case - Coplanar Forces

- 3 scalar equations

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum \tau_z = 0$$

(Axis of torque eq. is arbitrary)

Case I.

- Rigid body subject to two forces.
- Equilibrium if and only if the forces are equal in magnitude, opposite in direction and have the same line of action.

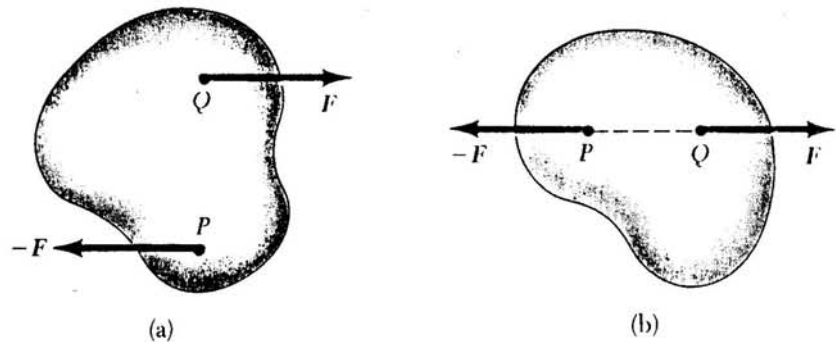


Figure (a) The body is not in equilibrium since the two forces do not have the same line of action. (b) The body is in equilibrium since the two forces act along the same line.

Case II.

- Rigid body subject to three forces
- If the body is in equilibrium, the lines of action of the three forces must intersect at a common point or the forces are parallel.

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0 \quad (\text{given})$$

$$\sum \vec{C}_S = 0$$

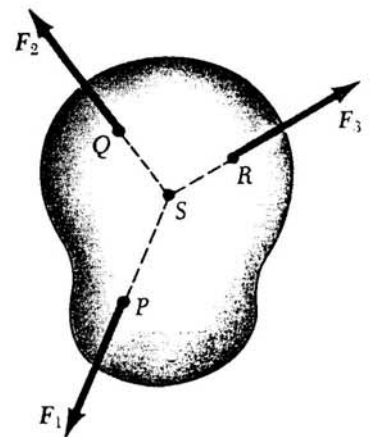


Figure If three forces act on a body that is in equilibrium, their lines of action must intersect at a point S.

### Torque Axis

- Consider a rigid object in translational equilibrium.

$$\therefore \sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots = 0$$

$\vec{r}_1 \rightarrow$  point of application of  $\vec{F}_1$   
 $\vdots$   
 etc.

The net torque about axis at O :

$$\sum \vec{\tau}_O = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \dots$$

Consider point O' having position vector  $\vec{r}'$  relative to point O.

$\vec{r}_1 - \vec{r}' \rightarrow$  point of application of  $\vec{F}_1$  rel. to O'.  
 $\vdots$   
 etc.

The torque about O' is :

$$\sum \vec{\tau}_{O'} = (\vec{r}_1 - \vec{r}') \times \vec{F}_1 + (\vec{r}_2 - \vec{r}') \times \vec{F}_2 + \dots$$

$$\sum \vec{\tau}_{O'} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \dots - \underbrace{\vec{r}' \times (\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots)}_{\equiv 0}$$

$$\therefore \sum \vec{\tau}_O = \sum \vec{\tau}_{O'}$$

If the body is in translational equilibrium and the net torque is zero about one point, it must be zero

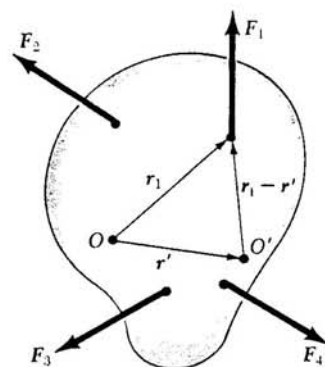


Figure Construction for showing that if the net torque about origin O is zero, the net torque about any other origin, such as O', must be zero.

about any other point. The other point can be inside or outside the boundaries of the object.

### Center of Gravity

In many equilibrium problems one of the forces acting is the weight of an object. The weight is distributed over the entire body.

In calculating torques due to the weight we will treat the entire weight of the body as being concentrated at a single point called the center of gravity. The center-of-gravity of a body coincides with its center-of-mass if the body is in a uniform gravitational field.

$$\vec{\omega}_1 = m_1 \vec{g} \quad (\text{Particle of mass } m_1)$$

$$\vec{\tau}_1 = \vec{r}_1 \times \vec{\omega}_1 \quad (\text{Torque about } O)$$

$$\sum \vec{\tau}_i = \vec{r}_1 \times m_1 \vec{g} + \vec{r}_2 \times m_2 \vec{g} + \dots$$

$$\sum \vec{\tau}_i = (m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots) \times \vec{g}$$

$$M = m_1 + m_2 + \dots \quad (\text{Total Mass})$$

$$\therefore \sum \vec{\tau}_i = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots}{M} \times M \vec{g}$$

$$\sum \vec{\tau}_i = \vec{R}_{cm} \times M \vec{g} = \vec{R}_{cm} \times \vec{W}$$

Total torque is the same as though the total weight acts at the position  $\vec{R}_{cm}$  of cm.

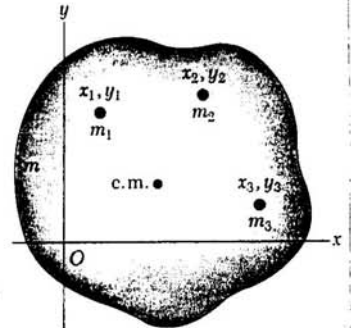


Figure A rigid body can be divided into many small particles with specific masses and coordinates. These can be used to locate the center of mass.

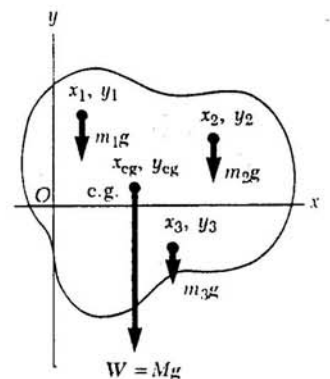


Figure The center of gravity of the rigid body is located at the center of mass if the value of  $g$  is constant over the body.

## Center-of-Mass [DEMO]

27-6

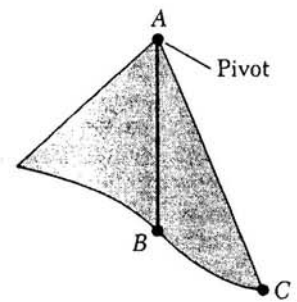
Objects are relatively small so that  $\vec{g}$  does not vary over the dimensions of the object. Equilibrium criteria can be used to find cm of irregularly shaped planar objects.

Suspend object successively from two different points.

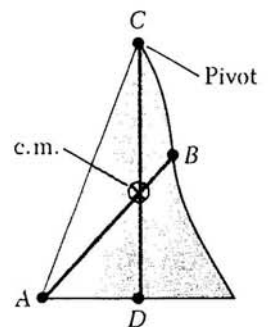
i) Suspend object from point A. A vertical line is drawn when object is in equilibrium. cm must be directly below A otherwise object would rotate.

ii) Suspend object from point C and draw a second vertical line CD. cm must be on this line.

Common intersection point is the desired center-of-mass



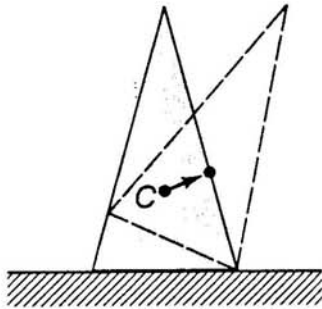
(a)



(b)

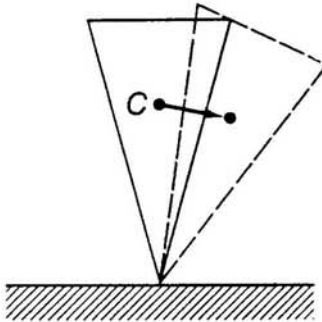
Figure An experimental technique for determining the center of mass of an irregular planar object. The object is hung freely from two different pivots, A and C. The intersection of the two vertical lines AB and CD locates the center of mass.

## Forms of Equilibrium



(a)

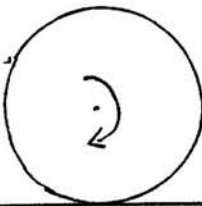
- Stable -



(b)

- Unstable -

**Fig.** (a) The cone is in a condition of stable equilibrium; a small tipping displacement *raises* the C.M. Upon release, the cone will return to its original position. (b) The cone is in a condition of unstable equilibrium; a small tipping displacement *lowers* the C.M. Upon release, the cone will fall.

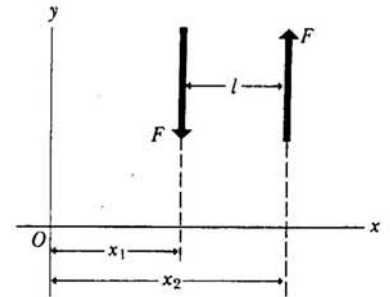


- Neutral -

Constant cm height as object moves.

## Couples

If two forces acting on a body have equal magnitude but opposite direction, with lines of action that are parallel but do not coincide — such a pair of forces is called a couple.



Two equal and opposite forces having different lines of action are called a *couple*. The torque of the couple is the same about all points and is equal to  $Fl$ .

Resultant:  $\vec{R} = \vec{F} - \vec{F} \equiv 0$ .

Torque about O:

$$\begin{aligned}\vec{\tau}_O &= x_2 F - x_1 F \\ &= (x_1 + l) F - x_1 F \\ &= l F\end{aligned}$$

The torque of the couple is the same about all points in the plane of the forces. It is equal to the product of the magnitude of either force and the perpendicular distance between them.

## Problem - Solving Strategy

27-9

1. Make a sketch of the body under consideration.
2. Draw a free-body diagram showing the forces acting on the body and no others. Do not include forces exerted by this body on other bodies. Guess directions of unknown force - do not worry about sign.
3. Draw coordinate axes and specify a positive sense of rotation for torques. Resolve forces into components with respect to axes you have chosen.
4. Write out equations expressing equilibrium conditions.  $\sum F_x = 0$ ,  $\sum F_y = 0$  and  $\sum \tau = 0$  are all separate equations. They must not be added!!!.
5. Choose convenient axes for calculating the net torque. Choice of origin is arbitrary. Note if the line of action of a force goes through a particular axis, the torque of the force about that axis is zero. Thus unknown forces or torques can be eliminated. Torques of forces can be computed as torques of force components.
6. May need to compute torques about several axes to have as many equations as unknowns. Solve the simultaneous equations for the unknowns.

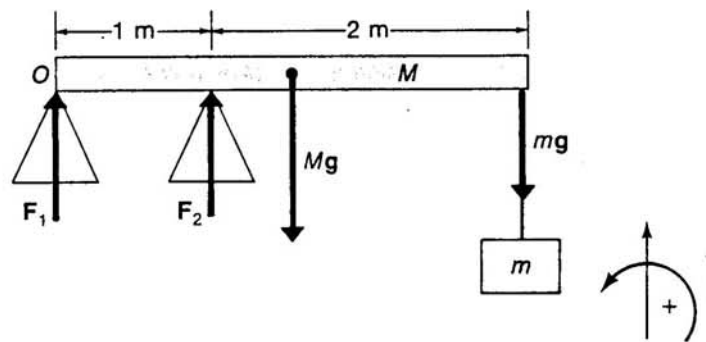
### Example: Pier Forces

27-11

$$m = 20 \text{ kg} \quad (\text{Load})$$

$$M = 50 \text{ kg} \quad (\text{Plank})$$

What forces  $F_1$  and  $F_2$   
do the piers exert  
on the plank?



$$\sum F_x = 0 \quad (\text{No Horizontal Forces})$$

$$\sum F_y = 0 \quad F_1 + F_2 - Mg - mg = 0$$

$$F_1 + F_2 = (M+m)g = (50+20)9.8 = 686 \text{ N}$$

Torques about 0 - line of action of  $\vec{F}_1$

$$\sum \tau_0 = 0 \quad F_1 \times (0) + F_2 (1) - Mg (1.5) - mg (3) = 0$$

$$F_2 = 50 \times 9.8 \times 1.5 + 20 \times 9.8 \times 3$$

$$F_2 = 1323 \text{ N}$$

$$F_1 = 686 - 1323$$
$$= -637 \text{ N}$$

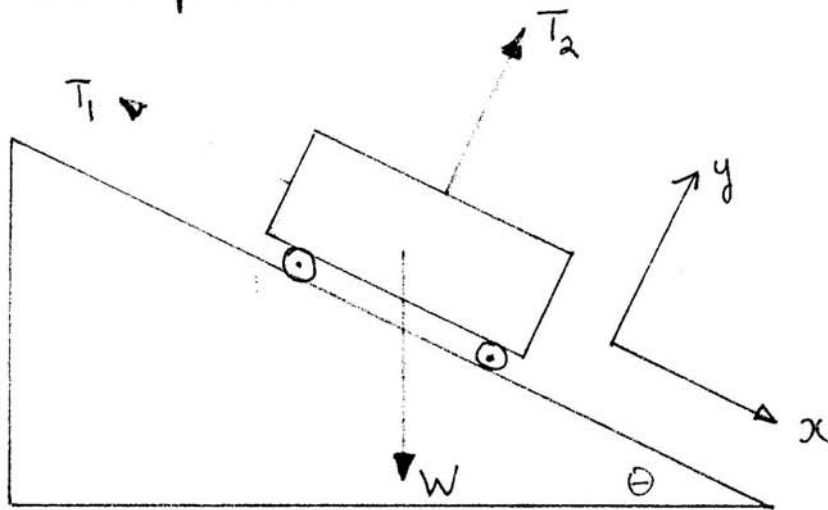
$F_1$  = downward, opposite to assumed direction.

Force on Pier #1 = tension

Force on Pier #2 = compression

### Example

27-13



$$W = 1020 \text{ g}$$

$$\theta = 30^\circ$$

$$-T_1 + W \sin \theta = 0$$

$$T_2 - W \cos \theta = 0$$

$$T_1 = W \sin \theta = 1020 \sin 30^\circ$$

$$T_1 = 510 \text{ g}$$

$$T_1^{\text{exp}} = 511$$

$$T_2 = W \cos \theta = 1020 \cos 30^\circ$$

$$T_2 = 883 \text{ g}$$

$$T_2^{\text{exp}} = 880$$

### Example [DEMO]

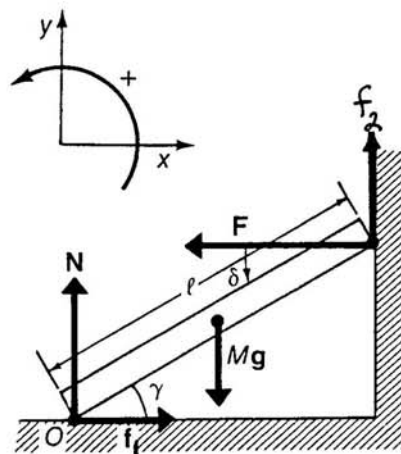
27-14

- Friction between plank + wall
- plank + floor
- $\mu_s$  (coefficient, static friction)

$$\sum F_x = 0 \quad f_1 - F = 0 \quad (1)$$

$$\sum F_y = 0 \quad N - Mg + f_2 = 0 \quad (2)$$

$$\sum \tau_o = 0 \quad lF \sin \gamma - \frac{1}{2} l Mg \cos \gamma + l f_2 \cos \gamma = 0 \quad (3)$$



sub (1) and (2) into (3)

$$f_1 \sin \gamma - \frac{Mg}{2} \cos \gamma + (Mg - N) \cos \gamma = 0 \quad (4)$$

$$2f_1 \tan \gamma + Mg - 2N = 0 \quad (5)$$

static force is maximum just before slipping.

$$f_1 = \mu_s N \quad f_2 = \mu_s F$$

$$\tan \gamma = \frac{2N - Mg}{2f_1} = \frac{N - Mg/2}{\mu_s N} \quad \text{From (5)}$$

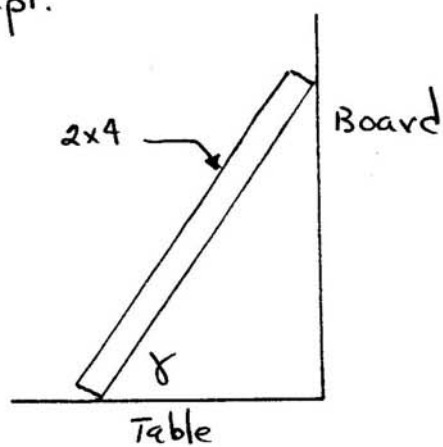
$$N = Mg - f_2 = Mg - \mu_s F = Mg - \mu_s f_1 = Mg - \mu_s^2 N \quad \text{From (2)}$$

$$\therefore N = \frac{Mg}{1 + \mu_s^2}$$

$$\tan \gamma = \frac{1 - Mg/2N}{\mu_s} = \frac{1 - \frac{1}{2}(1 + \mu_s^2)}{\mu_s} = \frac{1 - \mu_s^2}{2\mu_s}$$

$\mu_s$	$\tan \gamma$	$\gamma$
0.1	4.95	78.6°
0.2	2.40	67.4°
0.3	1.52	56.6°
0.4	1.05	46.4°
0.5	.75	36.9°
0.6	.53	28.1°
0.7	.36	20.0°
0.8	.23	12.7°

Expt.

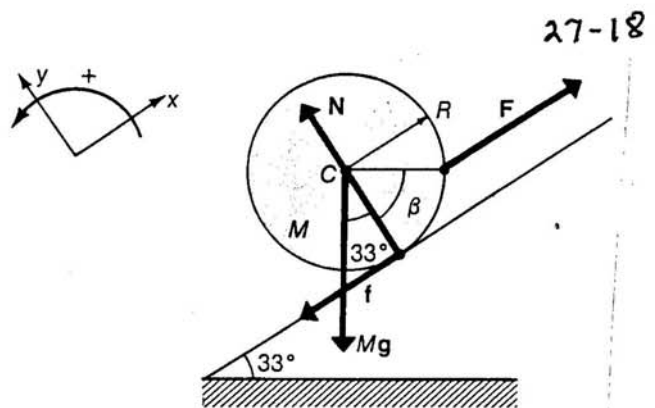


$$\gamma_{\text{Expt}} \sim 62^\circ$$

Example:

Sphere,  $M = 5 \text{ kg}$  is moved slowly up a plank by a string parallel to the plane.

$\mu_k = 0.42$ .



a) What is string tension?

b) What is angle  $\beta$  for attachment of string?

$$\sum F_y = 0 \quad N - Mg \cos 33^\circ = 0 \quad (1)$$

$$\sum F_x = 0 \quad F - f - Mg \sin 33^\circ = 0 \quad (2)$$

$$f = \mu_k N \quad (3)$$

$$(1) + (2) \quad F = Mg (\sin 33^\circ + 0.42 \cos 33^\circ)$$

$$F = 43.95 \text{ N}$$

Torques about C: - line of action of N and Mg.

$$\sum \tau_c = 0 \quad R(F \cos \beta) - fR = 0$$

$$\cos \beta = \frac{f}{F} = \frac{\mu_k Mg \cos 33^\circ}{F}$$

$$= 0.393$$

$$\beta = 66.9^\circ$$

### Example: Raising a Cylinder

Cylinder of weight  $W$  and radius  $R$  is to be raised onto a step of height  $h$ . A rope is wrapped around cylinder and pulled horizontally. No slipping.

What is minimum  $F$  and reaction force at  $P$ ?

When cylinder is ready to be raised, reaction force at  $Q$  goes to zero.  $\therefore$  Three forces on cylinder.

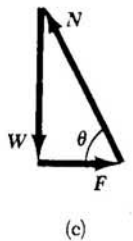
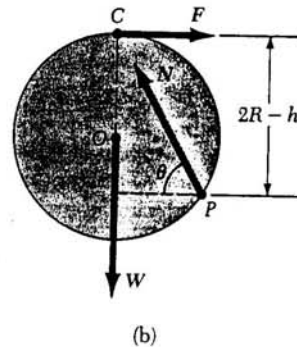
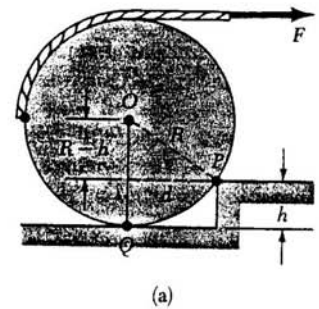


Figure (a) A cylinder of weight  $W$  being pulled by a force  $F$  over a step. (b) The free-body diagram for the cylinder when it is just ready to be raised. (c) The vector sum of the three external forces is zero.

$$d = \sqrt{R^2 - (R-h)^2} = \sqrt{2Rh - h^2} \quad (\text{moment arm of } \vec{W} \text{ about } P)$$

Torques about  $P$ :  $Wd - F(2R-h) = 0$

$$\therefore F = \frac{W \sqrt{2Rh - h^2}}{2R-h}$$

$$\sum F_x = 0$$

$$F - N \cos \theta = 0$$

$$\sum F_y = 0$$

$$N \sin \theta - W = 0$$

$$\tan \theta = \frac{W}{F} \quad \text{and} \quad N = \sqrt{W^2 + F^2}$$

$$W = 500 \text{ N}, \quad R = 0.3 \text{ m}, \quad h = 0.8 \text{ m}.$$

Solving:  $F = 385 \text{ N}$ ,  $\theta = 52.4^\circ$  and  $N = 631 \text{ N}$ .