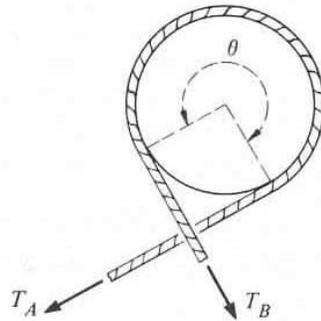


Problem 3: Experiment 5 Static Equilibrium Analysis

Part a) Suppose a rope of mass $m = 0.1\text{ kg}$ is connected at the same height to two walls and is allowed to hang under its own weight. At the contact point between the rope and the wall, the rope makes an angle $\theta = 10^\circ$ with respect to the horizontal. The object is to find the tension in the rope at the end and at the middle of the rope. In order to find the tension in the rope at the end and at the middle of the rope, you will need to think cleverly what to include as the system in your free body diagrams. Describe

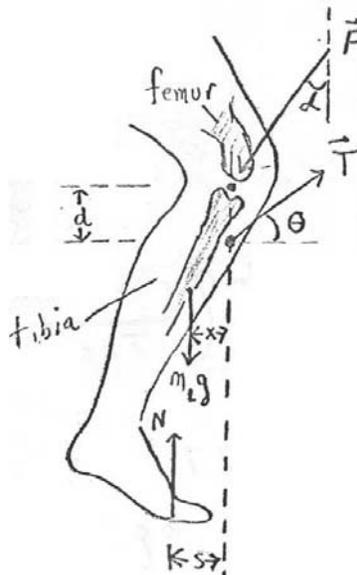
- What choice did you make for the system for your free body force diagram. Show all the forces acting on the diagram. What is the tension at the ends of the rope where it is connected to the walls?
- What choice did you make for the system for your free body force diagram. Show all the forces acting on the diagram. What is the tension in the rope at a point midway between the walls?

Part b) A device called a capstan is used aboard ships in order to control a rope which is under great tension. The rope is wrapped around a fixed drum, usually for several turns (the drawing below shows about three fourths turn as seen from overhead). The load on the rope pulls it with a force T_A , and the sailor holds it with a much smaller force T_B . Show that $T_B = T_A e^{-\mu_s \theta}$, where μ_s is the coefficient of static friction and θ is the total angle subtended by the rope on the drum? Here's a hint: choose a small mass element of arc length $R\Delta\theta$. Carefully indicate the forces acting on this section of the rope. Is the tension constant in this section?



Problem 4: Static Equilibrium The Knee

A man of mass $m = 70\text{ kg}$ is about to start a race. Assume the runner's weight is equally distributed on both legs. The patellar ligament in the knee is attached to the upper tibia and runs over the kneecap. When the knee is bent, a tensile force, \vec{T} , that the ligament exerts on the upper tibia, is directed at an angle of $\theta = 40^\circ$ with respect to the horizontal. The femur exerts a force \vec{F} on the upper tibia. The angle, α , this force makes with the vertical will vary and is one of the unknowns to solve for. Assume that the ligament is connected a distance, $d = 3.8\text{ cm}$, directly below the contact point of the femur on the tibia. The contact point between the foot and the ground is a distance $s = 3.6 \times 10^1\text{ cm}$ from the vertical line passing through contact point of the femur on the tibia. The center of mass of the lower leg lies a distance $x = 1.8 \times 10^1\text{ cm}$ from this same vertical line. Suppose the mass of the leg is a 1/5 of the mass of the body.



- Explain the origin of the forces that are shown in the above force diagram.
- What are the equations for static equilibrium of the forces?
- About point will you choose to analyze the torques? What are the equations for static equilibrium of the torques?
- Find the magnitude of the force \vec{T} of the patellar ligament on the tibia.
- Find the direction α of the force of the femur \vec{F} on the tibia.

Problem 3: Experiment 5 Static Equilibrium Analysis

Part a) Suppose a rope of mass $m = 0.1\text{ kg}$ is connected at the same height to two walls and is allowed to hang under its own weight. At the contact point between the rope and the wall, the rope makes an angle $\theta = 10^\circ$ with respect to the horizontal. The object is to find the tension in the rope at the end and at the middle of the rope. In order to find the tension in the rope at the end and at the middle of the rope, you will need to think cleverly what to include as the system in your free body diagrams. Describe

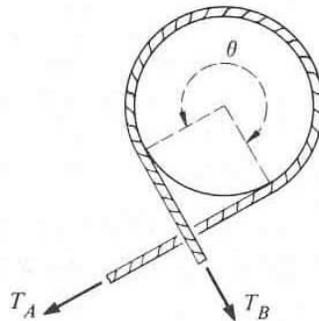
- What choice did you make for the system for your free body force diagram. Show all the forces acting on the diagram. What is the tension at the ends of the rope where it is connected to the walls?
- What choice did you make for the system for your free body force diagram. Show all the forces acting on the diagram. What is the tension in the rope at a point midway between the walls?

Answer:

a) Consider half of the rope as your system. The vertical component of the tension at the end of the rope must be equal to the weight of half of the rope so that $T_{end} = \frac{mg}{2\sin(\theta)} = 2.82\text{ N}$.

Then the tension at the middle must be equal to the horizontal component of the tension at the end that is $T_{middle} = \frac{mg}{2\tan(\theta)} = 2.78$

Part b) A device called a capstan is used aboard ships in order to control a rope which is under great tension. The rope is wrapped around a fixed drum, usually for several turns (the drawing below shows about three fourths turn as seen from overhead). The load on the rope pulls it with a force T_A , and the sailor holds it with a much smaller force T_B . Show that $T_B = T_A e^{-\mu_s \theta}$, where μ_s is the coefficient of static friction and θ is the total angle subtended by the rope on the drum? Here's a hint: choose a small mass element of arc length $R\Delta\theta$. Carefully indicate the forces acting on this section of the rope. Is the tension constant in this section?



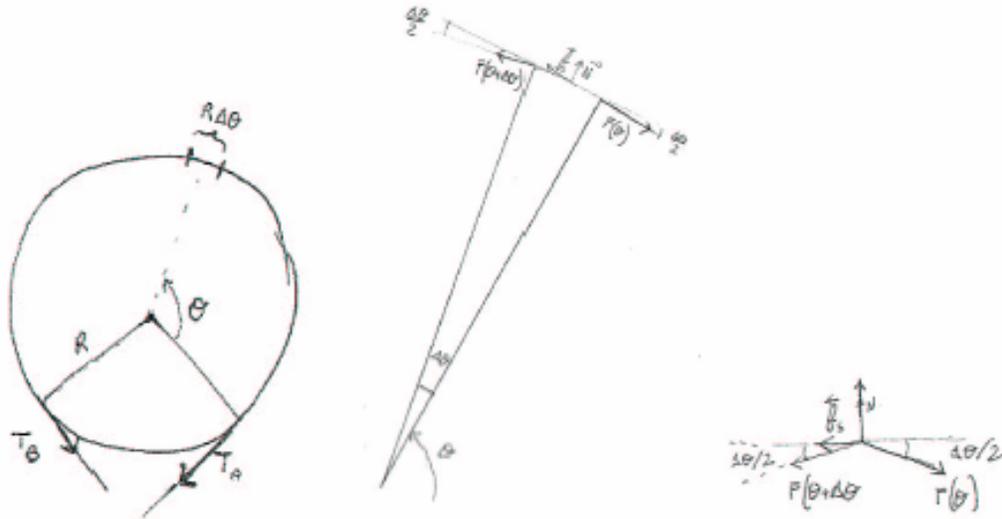
Answer:

b) Consider $\theta = 0$ where the extremity of the rope not pulled by the sailor touches the capstan. Then $T(0) = T_A$.

Consider now a section of the rope of length $R\Delta\theta$ at an angle θ on the capstan. The difference in tension between the two extremities of this piece is equal to $T(\theta + \Delta\theta) - T(\theta) = -\mu_s N$ where N is the normal force that the capstan exerts on the piece of rope: $N = 2T(\theta) \sin(\frac{\Delta\theta}{2}) \sim T(\theta)\Delta\theta$. So $\frac{d}{d\theta}T(\theta) = -\mu_s T(\theta)$ and solving:

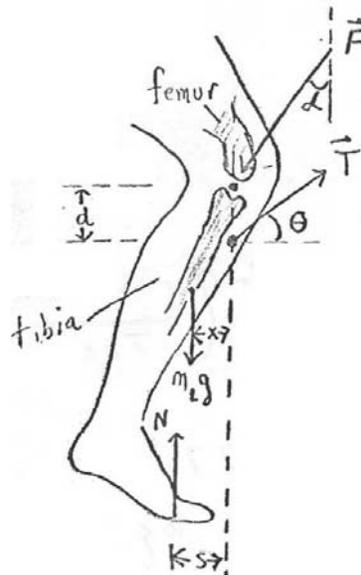
$$T(\theta) = T_A e^{-\mu_s \theta}$$

$$T(3\pi/2) = T_A e^{-3\pi\mu_s/2}$$



Problem 4: Static Equilibrium The Knee

A man of mass $m = 70\text{ kg}$ is about to start a race. Assume the runner's weight is equally distributed on both legs. The patellar ligament in the knee is attached to the upper tibia and runs over the kneecap. When the knee is bent, a tensile force, \vec{T} , that the ligament exerts on the upper tibia, is directed at an angle of $\theta = 40^\circ$ with respect to the horizontal. The femur exerts a force \vec{F} on the upper tibia. The angle, α , this force makes with the vertical will vary and is one of the unknowns to solve for. Assume that the ligament is connected a distance, $d = 3.8\text{ cm}$, directly below the contact point of the femur on the tibia. The contact point between the foot and the ground is a distance $s = 3.6 \times 10^1\text{ cm}$ from the vertical line passing through contact point of the femur on the tibia. The center of mass of the lower leg lies a distance $x = 1.8 \times 10^1\text{ cm}$ from this same vertical line. Suppose the mass of the leg is a 1/5 of the mass of the body.



- Explain the origin of the forces that are shown in the above force diagram.
- What are the equations for static equilibrium of the forces?
- About point will you choose to analyze the torques? What are the equations for static equilibrium of the torques?
- Find the magnitude of the force \vec{T} of the patellar ligament on the tibia.
- Find the direction α of the force of the femur \vec{F} on the tibia.

Answer:

Choose the unit vectors $\hat{\mathbf{i}}$ to point horizontally to the right and $\hat{\mathbf{j}}$ vertically upwards. The two conditions for static equilibrium are

(1) *The sum of the forces acting on the rigid body is zero,*

$$\vec{\mathbf{F}}_{total} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 + \vec{\mathbf{F}}_4 = \vec{\mathbf{0}}.$$

(2) *The vector sum of the torques about any point S in a rigid body is zero,*

$$\vec{\tau}_S^{total} = \vec{\tau}_{S,1} + \vec{\tau}_{S,2} + \vec{\tau}_{S,3} = \vec{\mathbf{0}}.$$

The first condition that the sum of the forces is zero becomes

$$\begin{aligned}\hat{\mathbf{i}}: -F \sin \alpha + T \cos \theta &= 0 \\ \hat{\mathbf{j}}: N - F \cos \alpha + T \sin \theta - (1/5)mg &= 0.\end{aligned}$$

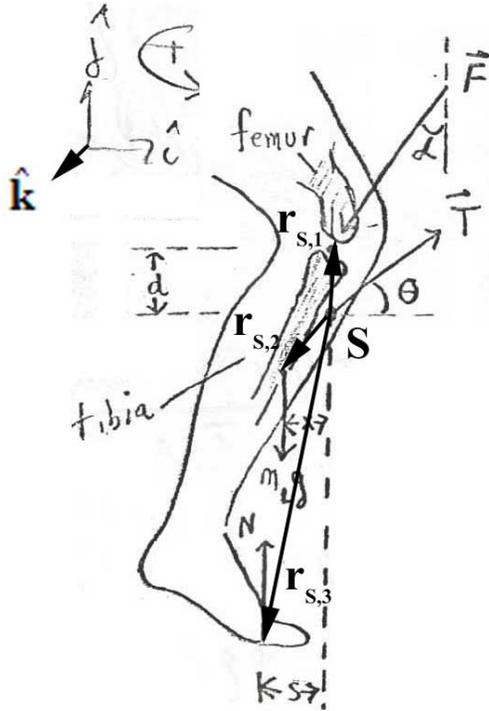
Since the weight is evenly distributed on the two feet, the normal force on one foot is equal to half the weight, or

$$N = (1/2)mg,$$

the second equation becomes

$$\hat{\mathbf{j}}: (1/2)mg - F \cos \alpha + T \sin \theta - (1/5)mg = 0$$

The torque-force diagram on the knee is shown below.



Choose the point of action of the ligament on the femur as the point S to compute the torque about. Note that the tensile force, \vec{T} , that the ligament exerts on the upper tibia will make no contribution to the torque about this point S . Choose counterclockwise c as the positive direction for the torque, this is the positive \hat{k} direction. Then the torque due to the force of the femur \vec{F} on the tibia is

$$\vec{\tau}_{S,1} = \vec{r}_{S,1} \times \vec{F} = d\hat{j} \times (-F \sin \alpha \hat{i} - F \cos \alpha \hat{j}) = dF \sin \alpha \hat{k} .$$

The torque due to the mass of the leg is

$$\vec{\tau}_{S,2} = \vec{r}_{S,2} \times -(1/5)mg\hat{j} = (-x\hat{i} - y_L\hat{j}) \times -(1/5)mg\hat{j} = (1/5)xmg\hat{k} .$$

The torque to the normal force of the ground is

$$\vec{\tau}_{S,3} = \vec{r}_{S,3} \times N\hat{j} = (-s\hat{i} - y_N\hat{j}) \times N\hat{j} = -sN\hat{k} = -(1/2)smg\hat{k} .$$

So the condition that the total torque about the point S vanishes becomes

$$\vec{\tau}_S^{total} = \vec{\tau}_{S,1} + \vec{\tau}_{S,2} + \vec{\tau}_{S,3} = \vec{0}$$

becomes

$$dF \sin \alpha \hat{\mathbf{k}} + (1/5)xmg\hat{\mathbf{k}} - (1/2)smg\hat{\mathbf{k}} = \vec{\mathbf{0}}.$$

Recalling our other two force equations

$$\hat{\mathbf{i}}: -F \sin \alpha + T \cos \theta = 0$$

$$\hat{\mathbf{j}}: (1/2)mg - F \cos \alpha + T \sin \theta - (1/5)mg = 0.$$

The horizontal force equation implies that

$$F \sin \alpha = T \cos \theta.$$

Substituting this into the torque equation yields

$$dT \cos \theta + (1/5)xmg - s(1/2)mg = 0$$

We can solve this equation for the magnitude of the force $\vec{\mathbf{T}}$ of the patellar ligament on the tibia,

$$T = \frac{s(1/2)mg - (1/5)xmg}{d \cos \theta}.$$

$$T = (70 \text{ kg})(9.8 \text{ m} \cdot \text{s}^{-2}) \frac{(3.6 \times 10^{-1} \text{ m})(1/2) - (1/5)(1.8 \times 10^{-1} \text{ m})}{(3.8 \times 10^{-2} \text{ m}) \cos(40^\circ)} = 3.4 \times 10^3 \text{ N}$$

We can now solve for the direction α of the force of the femur $\vec{\mathbf{F}}$ on the tibia as follows. Rewrite the two force equations as

$$F \cos \alpha = (1/2)mg + T \sin \theta - (1/5)mg = (3/10)mg + T \sin \theta$$

$$F \sin \alpha = T \cos \theta$$

Dividing these equations yields

$$\frac{F \cos \alpha}{F \sin \alpha} = \cot \alpha = \frac{(3/10)mg + T \sin \theta}{T \cos \theta}.$$

So

$$\alpha = \cot^{-1} \left(\frac{(3/10)mg + T \sin \theta}{T \cos \theta} \right)$$

$$\alpha = \cot^{-1} \left(\frac{(3/10)(70 \text{ kg})(9.8 \text{ m} \cdot \text{s}^{-2}) + (3.4 \times 10^3 \text{ N}) \sin(40^\circ)}{(3.4 \times 10^3 \text{ N}) \cos(40^\circ)} \right) = 4.7 \times 10^1 \text{ deg}$$

We can now use the horizontal force equation to calculate the magnitude $|\vec{F}|$ of the force of the femur \vec{F} on the tibia,

$$F = \frac{(3.4 \times 10^3 \text{ N}) \cos(40^\circ)}{\sin(4.7 \times 10^1 \text{ deg})} = 3.5 \times 10^3 \text{ N}$$

~~Problem 5: Post-Experiment 5a Analysis~~

You may work together as a group to solve this problem, but each group member should turn in a copy of this answer with problem set 05. If you like, just fill in the information above, complete this page, and attach it to your homework solutions.

Suppose a rope is tied rather tightly between two trees that are 30 m apart. You grab the middle of the rope and pull it perpendicular to the line between the trees with as much force as you can. Assume this force is 1000 N (about 225 lb), and the point where you are pulling on the rope is 1 m from the line joining the trees.

1. How much is the force tending to pull the trees together?

2. Give an example where you think this might be of practical use.