

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01T

Fall Term 2004

In-Class Problems 2 and 3: Projectile Motion Solutions

We would like each group to apply the problem solving strategy with the four stages (see below) to answer the following two problems.

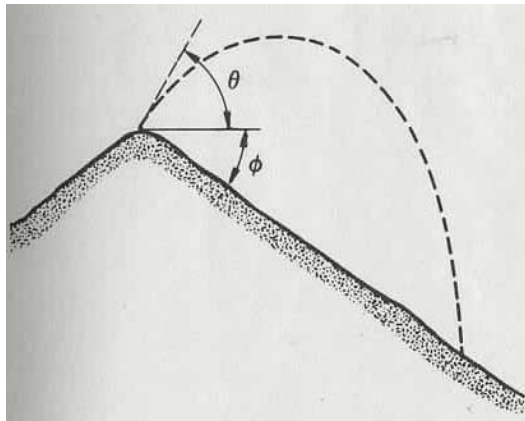
- I. Understand – get a conceptual grasp of the problem**
- II. Devise a Plan - set up a procedure to obtain the desired solution**
- III. Carry out your plan – solve the problem!**
- IV. Look Back – check your solution and method of solution**

For the first problem we have posed a series of questions for each of the above steps to help you learn how to use the problem solving strategy. We then leave space for your group to answer the question. You don't need to answer all these questions but they should help you approach the problem.

In the second problem, we invite you to try to solve it without help. If you would like some hints, we do pose a series of hints to consider.

In-Class Problem 2: Throwing a Stone Down a Hill

A person is standing on top of a hill which slopes downwards uniformly at an angle ϕ with respect to the horizontal. The person throws a stone at an initial angle θ_0 from the horizontal with an initial speed of v_0 . You may neglect air resistance. How far below the top of the hill does the stone strike the ground?

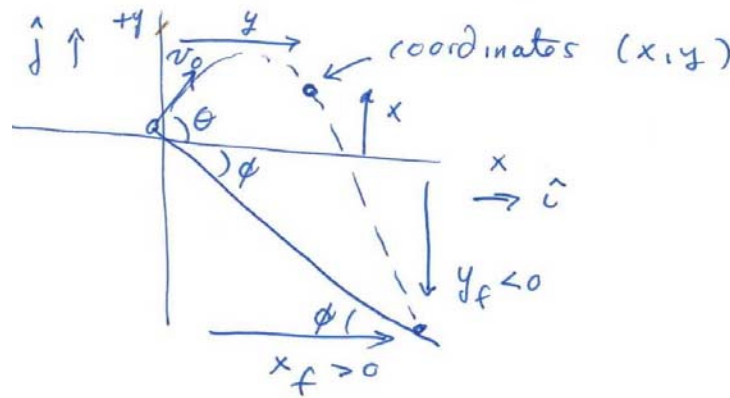


I. Understand – get a conceptual grasp of the problem

How would you model the horizontal and vertical motions of the stone? Draw a graph and a coordinate system. Where did you choose your origin? What choices did you make for positive axes? Do your choices for positive directions affect the signs for position, velocity, or acceleration? In terms of your coordinate system, is there any constraint condition for the horizontal and vertical position of the stone when it hits the ground in terms of the specified quantities in the problem (be careful with signs)?

Answer:

The problem involves constant velocity in the horizontal direction and constant acceleration a_y in the vertical direction.



I will choose a Cartesian coordinate system with the origin at the point the stone is thrown with vertical upwards positive and horizontal to the right as positive. Therefore $a_y = -g$. The key constraint condition is that the point where the stone strikes the hill has $y_f < 0$ and $x_f > 0$. Therefore $\tan \phi = y_f / x_f$ for $\phi < 0$ or $\tan \phi = -y_f / x_f$ for $\phi > 0$. I will choose $\phi > 0$.

II. Devise a Plan - set up a procedure to obtain the desired solution

In terms of your choices that you made in your model, what equations are applicable to this problem? Identify the different symbols that appear in your equation. Are they treated as knowns or unknowns? Do you have enough equations to solve the problem? Identify the quantity that you would like to solve for and design a strategy to find it.

Plan: I write down the horizontal and vertical equations for projectile motion

$$x(t) = x_0 + v_{x,0}t$$

$$y(t) = y_0 + v_{y,0}t - \frac{1}{2}gt^2$$

with the following initial and final conditions $x_0 = 0$, $v_{x,0} = v_0 \cos \theta_0$, $y_0 = 0$, and $v_{y,0} = v_0 \sin \theta_0$. The final conditions are $y(t_f) = y_f$ and $x(t_f) = x_f$.

Thus the equations of motion become

$$x_f = v_0 \cos \theta_0 t_f$$

$$y_f = v_0 \sin \theta_0 t_f - \frac{1}{2} g t_f^2$$

and the constraint

$$x_f = -y_f / \tan \phi.$$

III. Carry out your plan – solve the problem!

Be careful with signs!!!!

I am trying to solve for the final vertical position y_f . I will solve the horizontal equation for t_f yielding

$$t_f = x_f / v_0 \cos \theta_0.$$

I will introduce the constrain condition

$$t_f = -y_f / v_0 \cos \theta_0 \tan \phi$$

I will substitute this value for t_f into the vertical equation

$$y_f = v_0 \sin \theta_0 t_f - \frac{1}{2} g t_f^2 = v_0 \sin \theta_0 (-y_f / v_0 \cos \theta_0 \tan \phi) - \frac{1}{2} g (-y_f / v_0 \cos \theta_0 \tan \phi)^2.$$

This equation can be cleaned up

$$1 = -\tan \theta_0 / \tan \phi - \frac{1}{2} g y_f / v_0^2 \cos^2 \theta_0 \tan^2 \phi.$$

Now I can solve this for y_f

$$y_f = -\frac{2 v_0^2 \cos^2 \theta_0 \tan^2 \phi}{g} (1 + \tan \theta_0 / \tan \phi)$$

IV. Look Back – check your solution and method of solution

Does your solution make sense? Check special cases: what do you expect for an answer if the person is throwing the stone on level ground, or throwing the stone over a vertical cliff. Does your solution agree in these limits?

Answer: If $\phi = 0$ I need to be careful because this implies that $\tan \phi = 0$ so I cannot divide by $\tan \phi = 0$. Instead I need to note that $y_f = 0$. Then the equations of motion become

$$x_f = v_0 \cos \theta_0 t_f$$

$$0 = v_0 \sin \theta_0 t_f - \frac{1}{2} g t_f^2.$$

I can solve the second equation for t_f ,

$$t_f = 2v_0 \sin \theta_0 / g$$

and then for x_f

$$x_f = 2v_0^2 \cos \theta_0 \sin \theta_0 / g = 2v_0^2 \sin(2\theta_0) / g.$$

If $\phi = \pi/2$ then $\tan \pi/2 = \infty$ and

$$y_f = -\frac{2v_0^2 \cos^2 \theta_0 \tan^2 \phi}{g} (1 + \tan \theta_0 / \tan \phi) = -\frac{2v_0^2 \cos^2 \theta_0 \tan^2 \phi}{g} = -\infty.$$

The stone never hits the ground.

In-Class Problem 3: Hitting the Bucket

A person is standing on a ladder holding a pail. The person releases the pail from rest at a height h_1 above the ground. A second person standing a horizontal distance s_2 from the pail aims and throws a ball the instant the pail is released in order to hit the pail. The person throws the ball at a height h_2 above the ground, with an initial speed v_0 , and at an angle θ_0 with respect to the horizontal. You may ignore air resistance.

Questions:

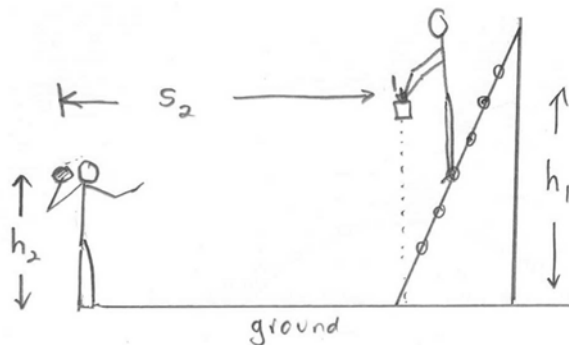
- Find an expression for the angle θ_0 that the person aims the ball in order to hit the pail as a function of the other variables given in the problem
- If the person aims correctly, find an expression for the range of speeds that the ball must be thrown at in order to hit the pail?

In case you need some help, try answering these questions.

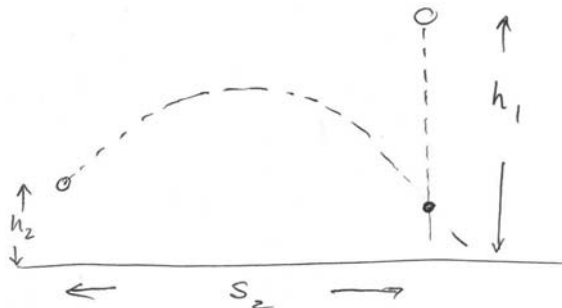
I. Understand – get a conceptual grasp of the problem

Sketch the motion of all the bodies in this problem. Introduce a coordinate system.

Sketch and Coordinate system:



Sketch of motion:

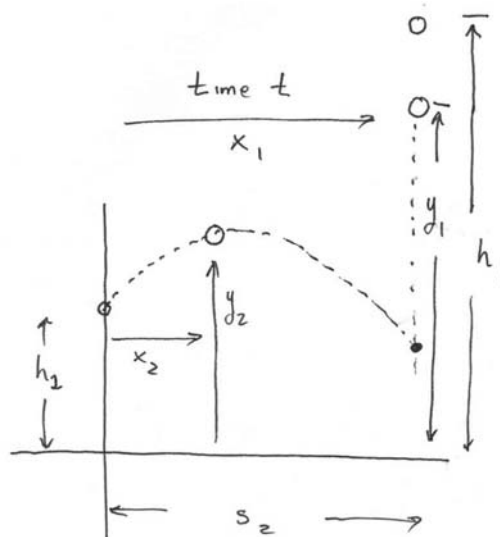


- c) Find an expression for the angle θ_0 that the person throws the ball as a function of h_1 , h_2 , and s_2 .
- d) Find an expression for the time of collision as a function of the initial speed of the ball v_0 , and the quantities h_1 , h_2 , and s_2 .
- e) Find an expression for the height above the ground where the collision occurred as a function of the initial speed of the ball v_0 , and the quantities h_1 , h_2 , and s_2 .
- f) Find an expression for the range of speeds (as a function of h_1 , h_2 , and s_2) that the ball can be thrown in order that the ball will collide with the pail?

There are two objects involved in this problem. Each object is undergoing free fall, so there is only one stage each. The pail is undergoing one dimensional motion. The ball is undergoing two dimensional motion. The parameters h_1 , h_2 , and s_2 are unspecified, so our answers will be functions of those symbolic expressions for the quantities.

Since the acceleration is unidirectional and constant, we will choose Cartesian coordinates, with one axis along the direction of acceleration. Choose the origin on the ground directly underneath the point where the ball is released. We choose upwards for the positive y-direction and towards the pail for the positive x-direction.

We choose position coordinates for the pail as follows. The horizontal coordinate is constant and given by $x_1 = s_2$. The vertical coordinate represents the height above the ground and is denoted by $y_1(t)$. The ball has coordinates $(x_2(t), y_2(t))$. We show these coordinates in the figure below.



II. Devise a Plan - set up a procedure to obtain the desired solution

Question: What equations of motion follow from your model for the position and velocity functions of each body?

Model: The pail undergoes constant acceleration $(a_y)_1 = -g$ in the vertical direction downwards and the ball undergoes uniform motion in the horizontal direction and constant acceleration downwards in the vertical direction, with $(a_x)_2 = 0$ and $(a_y)_2 = -g$.

Equations of Motion for Pail:

The initial conditions for the pail are $(v_{y,0})_1 = 0$, $x_1 = s_2$, $(y_0)_1 = h_1$. Since the pail moves vertically, the pail always satisfies the constraint condition $x_1 = s_2$ and $v_{x,1} = 0$. The equations for position and velocity of the pail simplify to

$$y_1(t) = h_1 - \frac{1}{2}gt^2$$

$$v_{y,1}(t) = -gt$$

Equations of Motion for Ball:

The initial position is given by $(x_0)_2 = 0$, $(y_0)_2 = h_2$. The components of the initial velocity are given by $(v_{y,0})_2 = v_0 \sin(\theta_0)$ and $(v_{x,0})_2 = v_0 \cos(\theta_0)$, where v_0 is the magnitude of the initial velocity and θ_0 is the initial angle with respect to the horizontal. So the equations for position and velocity of the ball simplify to

$$x_2(t) = v_0 \cos(\theta_0)t$$

$$v_{x,2}(t) = v_0 \cos(\theta_0)$$

$$y_2(t) = h_2 + v_0 \sin(\theta_0)t - \frac{1}{2}gt^2$$

$$v_{y,2}(t) = v_0 \sin(\theta_0) - gt$$

Question: How many independent equations and unknowns do you have? Should the quantities h_1 , h_2 , and s_2 be treated as knowns or unknowns.

Answer: Note that the quantities h_1 , h_2 , and s_2 should be treated as known quantities although no numerical values were given, only symbolic expressions. There are six independent equations with 9 as yet unspecified quantities $y_1(t)$, t , $y_2(t)$, $x_2(t)$, $v_{y,1}(t)$, $v_{y,2}(t)$, $v_{x,2}(t)$, v_0 , θ_0 .

$$y_1(t) = h_1 - \frac{1}{2}gt^2$$

$$v_{y,1}(t) = -gt$$

$$x_2(t) = v_0 \cos(\theta_0)t$$

$$v_{x,2}(t) = v_0 \cos(\theta_0)$$

$$y_2(t) = h_2 + v_0 \sin(\theta_0)t - \frac{1}{2}gt^2$$

$$v_{y,2}(t) = v_0 \sin(\theta_0) - gt$$

So we need two more conditions, in order to find expressions for the initial angle, θ_0 , the time of collision, t_a , and the spatial location of the collision point specified by $y_1(t_a)$ or $y_2(t_a)$ in terms of the one unspecified parameter v_0 .

Question: What mathematical formulae follow from the phase “hits the pail”?

Answer: At the collision time $t = t_a$, the collision occurs when the two balls are located at the same position. Therefore

$$y_1(t_a) = y_2(t_a), \text{ and } x_2(t_a) = x_1 = s_2.$$

Question: Clean up your equations. What strategy can you design for finding the angle the second person needs to aim the ball?

Answer:

We shall apply the conditions we found for the ball hitting the pail.

$$h_1 - \frac{1}{2}gt_a^2 = h_2 + v_0 \sin(\theta_0)t_a - \frac{1}{2}gt_a^2$$

$$s_2 = v_0 \cos(\theta_0)t_a$$

From the first equation, the term $\frac{1}{2}gt_a^2$ cancels from both sides. Therefore we have that

$$h_1 = h_2 + v_0 \sin(\theta_0)t_a$$

$$s_2 = v_0 \cos(\theta_0)t_a.$$

We will now solve these equations for $\tan(\theta_0) = \sin(\theta_0)/\cos(\theta_0)$, and thus the angle the person throws the ball in order to hit the pail.

III. Carry out your plan – solve the problem!

We rewrite these equations as

$$v_0 \sin(\theta_0)t_a = h_1 - h_2$$

$$v_0 \cos(\theta_0)t_a = s_2$$

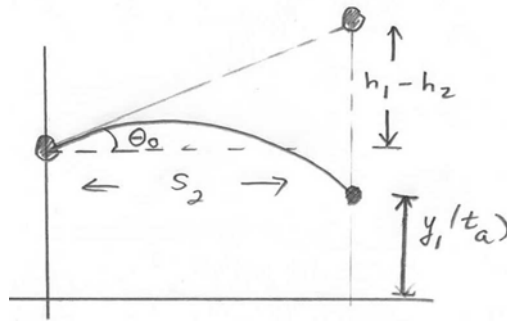
Dividing these equations yields

$$\frac{v_0 \sin(\theta_0)t_a}{v_0 \cos(\theta_0)t_a} = \tan(\theta_0) = \frac{h_1 - h_2}{s_2}.$$

So the initial angle is independent of v_0 , and is given by

$$\theta_0 = \tan^{-1}((h_1 - h_2)/s_2)$$

From the figure below we can see that $\tan(\theta_0) = \frac{h_1 - h_2}{s_2}$, implies that the second person aims the ball at the initial position of the pail.



Question: When does the ball collide with the pail?

Answer: In order to find the time of collision as a function of the initial speed, we begin with our results that

$$v_0 \sin(\theta_0) t_a = h_1 - h_2$$

$$v_0 \cos(\theta_0) t_a = s_2$$

We square both of the equations above and utilize the trigonometric identity

$$\sin^2(\theta_0) + \cos^2(\theta_0) = 1.$$

So our squared equations become

$$v_0^2 \sin^2(\theta_0) t_a^2 = (h_1 - h_2)^2$$

$$v_0^2 \cos^2(\theta_0) t_a^2 = s_2^2$$

Adding these equations together yields

$$v_0^2 \sin^2(\theta_0) t_a^2 + v_0^2 \cos^2(\theta_0) t_a^2 = v_0^2 t_a^2 = s_2^2 + (h_1 - h_2)^2.$$

We can solve this for the time of collision

$$t_a = \left(\frac{s_2^2 + (h_1 - h_2)^2}{v_0^2} \right)^{1/2}.$$

Question: At what height does the ball collide with the pail?

Answer: We can use the y-coordinate function of either the ball or the pail at $t = t_a$. Since it had no initial y velocity, it's easier to use the pail,

$$y_1(t_a) = h_1 - \frac{g(s_2^2 + (h_1 - h_2)^2)}{2v_0^2}$$

Question: What is the minimum speed the person must throw the ball in order to ensure that there will be a collision?

Answer: Suppose the ball and the pail collide exactly at the ground at the time $t = t_b$. The condition is that the speed v_0 must be great enough such that the ball reaches $x_2(t_b) = s_2$ before the ball hits the ground, $y_2(t_b) = y_1(t_b) = 0$.

So we require that

$$x_2(t_b) = v_0 \cos(\theta_0) t_b \geq s_2$$

This condition is easiest to apply when solving for the time, $t = t_b$, that the pail hits the ground,

$$y_1(t_b) = h_1 - \frac{1}{2} g t_b^2 = 0.$$

Thus

$$t_b = \left(\frac{2h_1}{g} \right)^{1/2}.$$

Therefore the condition becomes

$$x_2(t_b) = v_0 \cos(\theta_0) \left(\frac{2h_1}{g} \right)^{1/2} \geq s_2$$

or

$$v_0 \geq \left(\frac{g}{2h_1} \right)^{1/2} \frac{s_2}{\cos(\theta_0)} = \left(\frac{g}{2h_1} \right)^{1/2} \frac{s_2}{\cos(\tan^{-1}((h_1 - h_2)/s_2))}.$$

IV. Look Back – check your solution and method of solution

Question: check your algebra and your units. Any obvious errors?

Answer: No obvious errors.

Question: Can you see that the answer is correct now that you have it – often simply by retrospective inspection?

Answer: The person aims at the pail at the point where the pail was released. Both undergo free fall so the key result was that the vertical position obeys

$$h_1 - \frac{1}{2}gt_a^2 = h_2 + v_0 \sin(\theta_0)t_a - \frac{1}{2}gt_a^2.$$

The distance traveled due to gravitational acceleration are the same for both so all that matters is the contribution from the initial positions and the vertical component of velocity

$$h_1 = h_2 + v_0 \sin(\theta_0)t_a.$$

Since the time is related to the horizontal distance by

$$s_2 = v_0 \cos(\theta_0)t_a$$

This is now as if both objects were moving at constant velocity.

Question: Substitute some values for the initial conditions that test the limits of your answer. Make the pail very far away or very close and what your answer predicts about the time of flight or the collision height.

Answer: Let $s_2 = 10 \text{ m}$, $v_0 = 20 \text{ m} \cdot \text{s}^{-1}$, let $h_2 = 2 \text{ m}$, and $h_1 = 4 \text{ m}$. Then the condition for the initial speed is satisfied since

$$\begin{aligned} v_0 &\geq \left(\frac{g}{2h_1} \right)^{1/2} \frac{s_2}{\cos(\tan^{-1}((h_1 - h_2)/s_2))} \\ &= \left(\frac{10 \text{ m} \cdot \text{s}^{-2}}{2(4 \text{ m})} \right)^{1/2} \frac{10 \text{ m}}{\cos(\tan^{-1}((4 \text{ m} - 2 \text{ m})/10 \text{ m}))} = 11.4 \text{ m} \cdot \text{s}^{-1} \end{aligned}$$

Question: Can you solve it a different way? Is the problem equivalent to one you've solved before if the variables have some specific values?

Answer: This is an unusual application of moving to a reference frame accelerating downwards with $A_y = -g$. Then the problem is simply

$$y_1'(t) = h_1$$

$$x_2'(t) = v_0 \cos(\theta_0) t$$

$$y_2'(t) = h_2 + v_0 \sin(\theta_0) t$$