

Lecture 5. September 16, 2005

Homework. Problem Set 2 Part I: (a)–(e); Part II: Problem 2.

Practice Problems. Course Reader: 1I-1, 1I-4, 1I-5

1. Example of implicit differentiation. Let $y = f(x)$ be the unique function satisfying the equation,

$$\frac{1}{x} + \frac{1}{y} = 2.$$

What is slope of the tangent line to the graph of $y = f(x)$ at the point $(x, y) = (1, 1)$?

Implicitly differentiate each side of the equation to get,

$$\frac{d}{dx} \left(\frac{1}{x} \right) + \frac{d}{dx} \left(\frac{1}{y} \right) = \frac{d(2)}{dx} = 0.$$

Of course $(1/x)' = (x^{-1})' = -x^{-2}$. And by the rule $d(u^n)/dx = nu^{n-1}(du/dx)$, the derivative of $1/y$ is $-y^{-2}(dy/dx)$. Thus,

$$-x^{-2} - y^{-2} \frac{dy}{dx} = 0.$$

Plugging in x equals 1 and y equals 1 gives,

$$-1 - 1y'(1) = 0,$$

whose solution is,

$$y'(1) = -1.$$

In fact, using that $1/y$ equals $2 - 1/x$, this can be solved for every x ,

$$\frac{dy}{dx} = (x^{-2})/(y^{-2}) = \frac{1}{x^2} \cdot \frac{1}{(2 - 1/x)^2} = \frac{1}{(2x - 1)^2}.$$

2. Rules for exponentials and logarithms. Let a be a positive real number. The basic rules of exponentials are as follows.

Rule 1. If a^b equals B and a^c equals C , then a^{b+c} equals $B \cdot C$, i.e.,

$$a^{b+c} = a^b \cdot a^c.$$

Rule 2. If a^b equals B and B^d equals D , then a^{bd} equals D , i.e.,

$$(a^b)^d = a^{bd}.$$

If a^b equals B , the *logarithm with base a* of B is defined to be b . This is written $\log_a(B) = b$. The function $B \rightarrow \log_a(B)$ is defined for all positive real numbers B . Using this definition, the rules of exponentiation become rules of logarithms.

Rule 1. If $\log_a(B)$ equals b and $\log_a(C)$ equals c , then $\log_a(B \cdot C)$ equals $b + c$, i.e.,

$$\log_a(B \cdot C) = \log_a(B) + \log_a(C).$$

Rule 2. If $\log_a(B)$ equals b and B^d equals D , then $\log_a(D)$ equals $d \log_a(B)$, i.e.,

$$\log_a(B^d) = d \log_a(B).$$

Rule 3. Since $\log_B(D)$ equals d , an equivalent formulation is $\log_a(D)$ equals $\log_a(B) \log_B(D)$, i.e.,

$$\log_a(D) = \log_a(B) \log_B(D).$$

3. The derivative of a^x . Let a be a positive real number. What is the derivative of a^x ? Denote the derivative of a^x at $x = 0$ by $L(a)$. It equals the value of the limit,

$$L(a) = \lim_{h \rightarrow 0} \frac{a^h - 1}{h}.$$

Then for every x_0 , the derivative of a^x at x_0 equals,

$$\lim_{h \rightarrow 0} \frac{a^{x_0+h} - a^{x_0}}{h}.$$

By Rule 1, a^{x_0+h} equals $a^{x_0}a^h$. Thus the limit factors as,

$$\lim_{h \rightarrow 0} \frac{a^{x_0}a^h - a^{x_0}}{h} = a^{x_0} \lim_{h \rightarrow 0} \frac{a^h - 1}{h}.$$

Therefore, for every x , the derivative of a^x is,

$$\frac{d(a^x)}{dx} = L(a)a^x.$$

What is $L(a)$? To figure this out, consider how $L(a)$ changes as a changes. First of all,

$$L(a^b) = \lim_{h \rightarrow 0} \frac{(a^b)^h - 1}{h}.$$

By Rule 2, $(a^b)^h$ equals a^{bh} . So the limit is,

$$L(a^b) = \lim_{h \rightarrow 0} \frac{a^{bh} - 1}{h} = b \lim_{h \rightarrow 0} \frac{a^{bh} - 1}{bh}.$$

Now, inside the limit, make the substitution that k equals bh . As h approaches 0, also k approaches 0. So the limit is,

$$L(a^b) = b \lim_{k \rightarrow 0} \frac{a^k - 1}{k} = bL(a).$$

This is very similar to Rule 2 for logarithms.

Choose a number a_0 bigger than 1, say $a_0 = 2$. Then for every positive real number a , $a = a_0^b$ where $b = \log_{a_0}(a)$. Thus,

$$L(a) = L(a_0^b) = bL(a_0) = L(a_0) \log_{a_0}(a).$$

So, with a_0 fixed and a allowed to vary, $L(a)$ is just the logarithm function $\log_{a_0}(a)$ scaled by $L(a_0)$. Looking at the graph of $(a_0)^x$, it is geometrically clear that $L(a_0)$ is positive (though we have not *proved* that $L(a_0)$ is even defined). Thus the graph of $L(a)$ looks qualitatively like the graph of $\log_{a_0}(a)$. In particular, for a less than 1, $L(a)$ is negative. The value $L(1)$ equals 0. And $L(a)$ approaches $+\infty$ and a increases. Therefore, there must be a number where L takes the value 1. By long tradition, this number is called e ;

$$L(e) = \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1.$$

This is the definition of e . It sheds very little light on the decimal value of e .

Because e is so important, the logarithm with base e is given a special name: the **natural logarithm**. It is denoted by,

$$\ln(a) = \log_e(a).$$

So, finally, $L(a)$ equals,

$$L(a) = \log_e(a)L(e) = \ln(a)(1) = \ln(a).$$

This leads to the formula for the derivative of a^x ,

$$\frac{d(a^x)}{dx} = \ln(a)a^x.$$

In particular,

$$\frac{d(e^x)}{dx} = e^x.$$

In fact, e^x is characterized by the property above and the property that e^0 equals 1.

~~4. The derivative of $\log_a(x)$ and the value of e .~~ By the chain rule,

$$\frac{d(a^u)}{dx} = \ln(a)a^u \frac{du}{dx}.$$

~~For $u = \log_a(x)$, a^u equals x . Thus,~~

$$\frac{d(a^u)}{dx} = \frac{d(x)}{dx} = 1.$$

~~Thus,~~

$$\ln(a)a^u \frac{du}{dx} = 1.$$