

Name: _____

Problem 3: _____ /30

Problem 3(30 points) A box is made from two identical square sheets of metal with edge length E . A small square of edge length x is removed and the two flaps are folded up. Find the value of x that maximizes the volume of the box.

Solution to (f) By the product rule, the second derivative of $f(x)$ is,

$$\frac{d}{dx} \left(\frac{x^{-3/2}}{2} (x-1) \right) = \frac{1}{2} \frac{d(x^{-3/2})}{dx} (x-1) + \frac{1}{2} x^{-3/2} \frac{d(x-1)}{dx} = \frac{-3}{4} x^{-5/2} (x-1) + \frac{1}{2} x^{-3/2}.$$

This simplifies to,

$$\frac{d^2 f}{dx^2} = \frac{x^{-5/2}}{4} (-x+3).$$

This is positive for $0 < x < 3$, negative for $x > 3$ and zero for $x = 3$. Notice $f(3) = \sqrt{3} + 1/\sqrt{3} = \frac{4\sqrt{3}}{3}$. Therefore,

there is a unique inflection point with coordinates $(3, 4\sqrt{3}/3)$.

(g) (7 points) On the number line, identify where $f''(x)$ is positive, negative or zero.

Solution to (g) By the previous part,

$f''(x)$ is positive for $0 < x < 3$, negative for $x > 3$ and zero for $x = 3$.

(h) (10 points) On the grid given, sketch the graph of $y = f(x)$.

Solution to (h) The sketch of the graph is given in **Figure 1**. The local minimum is the green dot, and the inflection point is the red star.

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Solution to Problem 3 The length and width of the base of the box are each $l = E - x$ and $w = E - x$. The height of the box is $h = x$. The volume of the box is,

$$V = lwh = (E - x)^2 x.$$

Therefore,

$$\frac{dV}{dx} = 2(E - x)(-1)x + (E - x)^2 = (E - x)(-2x + (E - x)) = (E - x)(E - 3x).$$

The critical points are $x = E/3$ and $x = E$. The endpoints are $x = 0$ and $x = E$. Since $V(0) = 0$ and $V(E) = 0$, the maximum volume occurs when $x = E/3$, giving

$$V = 4E^3/27 \text{ when } x = E/3.$$

Problem 4 (10 points) Find the quadratic approximation of $\sqrt{2 - \cos(\theta)}$ for $\theta \approx 0$.