

~~Problem 2~~(10 points) Using the definition of the Riemann integral, upper sums, and the formulas for special sums in the textbook, compute that

$$\int_{-a}^a (a-x)(a+x)dx = \frac{4a^3}{3},$$

~~for all  $a > 0$ .~~

**Problem 3**(15 points) This problem is a special case of a more general method for computing antiderivatives that will be developed systematically later in the semester.

(a)(10 points) Compute the antiderivative  $\int \frac{1}{1-u^2} du$  as follows. Consider the sum  $\frac{a}{u+1} + \frac{b}{u-1}$ , where  $a$  and  $b$  are unspecified constants, and clear denominators. Find a choice of  $a$  and  $b$  so the sum simplifies to  $\frac{1}{1-u^2}$ . Now replace  $\frac{1}{1-u^2}$  in the integrand by your expression  $\frac{a}{u+1} + \frac{b}{u-1}$ . Use  $\ln(u+1)$ ,  $\ln(u-1)$  to compute the antiderivative.

(b)(5 points) Reduce the computation of the antiderivative  $\int \frac{1}{\cos(x)} dx$  to (a) by multiplying numerator and denominator of the integrand by  $\cos(x)$ , using the trig identity  $(\sin(x))^2 + (\cos(x))^2 = 1$  and substituting  $u = \sin(x)$ . Don't forget to back-substitute. Also, say for what range of  $x$  your antiderivative is valid ( $\ln(y)$  isn't defined if  $y$  is negative!).

By the fundamental theorem of calculus, we have  $\frac{dg}{du} = e^{-u^2} = e^{-x}$ . Therefore

$$\frac{d}{dx} \int_0^{\sqrt{x}} e^{-t^2} dt = x^{-1/2} e^{-x} / 2 \quad (3)$$

**Problem 2**(10 points) Using the definition of the Riemann integral, upper sums, and the formulas for special sums in the textbook, compute that

$$\int_{-a}^a (a-x)(a+x) dx = \frac{4a^3}{3},$$

for all  $a > 0$ .

**Solution to Problem 2** Since the function  $f(x) = (a-x)(a+x) = a^2 - x^2$  is even, we can rewrite the given integral as

$$\int_{-a}^a (a-x)(a+x) dx = \int_{-a}^a (a^2 - x^2) dx = 2 \int_0^a (a^2 - x^2) dx$$

We partition the interval  $[0, a]$  uniformly, i.e. by  $x_k = \frac{ka}{n}$ , into  $n$  equal subintervals, each of which has length  $\frac{a}{n}$ . Remembering that we are required to compute the upper sums, we calculate

$$y_k = \max_{(k-1)\frac{a}{n} \leq x \leq k\frac{a}{n}} (a^2 - x^2) = a^2 - \frac{k^2 a^2}{n^2} = \frac{a^2}{n^2} (n^2 - k^2).$$

Therefore

$$\begin{aligned} A_{max} &= \sum_{k=1}^n \frac{a^2}{n^2} (n^2 - k^2) \frac{a}{n} = \frac{a^3}{n^3} \sum_{k=1}^n (n^2 - k^2) \\ &= \frac{a^3}{n^3} \left( n^2 n - \frac{n(n+1)(2n+1)}{6} \right) = a^3 \left[ 1 - \frac{1}{6} \left( 2 + \frac{1}{n} \right) \left( 1 + \frac{1}{n} \right) \right] \xrightarrow{n \rightarrow \infty} a^3 \left( 1 - \frac{2}{6} \right) = \frac{2a^3}{3} \end{aligned}$$

Hence,

$$\int_{-a}^a (a-x)(a+x) dx = 2 \int_0^a (a^2 - x^2) dx = 2 \left( \frac{2a^3}{3} \right) = \frac{4a^3}{3}.$$

**Problem 3**(15 points) This problem is a special case of a more general method for computing antiderivatives that will be developed systematically later in the semester.

(a)(10 points) Compute the antiderivative  $\int \frac{1}{1-u^2} du$  as follows. Consider the sum  $\frac{a}{u+1} + \frac{b}{u-1}$ , where  $a$  and  $b$  are unspecified constants, and clear denominators. Find a choice of  $a$  and  $b$  so the sum simplifies to  $\frac{1}{1-u^2}$ . Now replace  $\frac{1}{1-u^2}$  in the integrand by your expression  $\frac{a}{u+1} + \frac{b}{u-1}$ . Use  $\ln(u+1)$ ,  $\ln(u-1)$  to compute the antiderivative.

**Solution to (a)** Following the hint, we let

$$\frac{1}{1-u^2} = \frac{a}{1+u} + \frac{b}{1-u} = \frac{a(1-u) + b(1+u)}{1-u^2} = \frac{(-a+b)u + (a+b)}{1-u^2}$$

Therefore, we must have

$$-a + b = 0, \quad a + b = 1$$

Solving this set of equations, we obtain  $a = b = 1/2$ . In summary,

$$\frac{1}{1-u^2} = \frac{1}{2} \frac{1}{1+u} + \frac{1}{2} \frac{1}{1-u} = \frac{1}{2} \frac{1}{1+u} - \frac{1}{2} \frac{1}{u-1}$$

Therefore

$$\begin{aligned} \int \frac{1}{1-u^2} du &= \int \frac{1}{2} \frac{1}{1+u} du - \int \frac{1}{2} \frac{1}{u-1} du \\ &= \frac{1}{2} \ln(|1+u|) - \frac{1}{2} \ln(|u-1|) = \frac{1}{2} \ln\left(\frac{|u+1|}{|u-1|}\right) = \ln(\sqrt{|u+1|}/\sqrt{|u-1|}). \end{aligned}$$

**(b)**(5 points) Reduce the computation of the antiderivative  $\int \frac{1}{\cos(x)} dx$  to **(a)** by multiplying numerator and denominator of the integrand by  $\cos(x)$ , using the trig identity  $(\sin(x))^2 + (\cos(x))^2 = 1$  and substituting  $u = \sin(x)$ . Don't forget to back-substitute. Also, say for what range of  $x$  your antiderivative is valid ( $\ln(y)$  isn't defined if  $y$  is negative!).

**Solution to (b)** Following the hint,

$$\int \frac{1}{\cos(x)} dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{\cos x}{1 - \sin^2 x} dx \quad (4)$$

We let  $u = \sin x$ , then  $du = \cos x dx$ , and the indefinite integral (4) becomes

$$\int \frac{du}{1-u^2} = \frac{1}{2} \ln\left(\frac{|u+1|}{|u-1|}\right) = \ln(\sqrt{|1+\sin x|}/\sqrt{|1-\sin x|}).$$

Because of the absolute values, the argument of the logarithm never gets negative. However, it still vanishes when  $1 + \sin x = 0$ , (i.e. when  $x = -\frac{\pi}{2} + 2\pi n$  for integers  $n$ ) and still becomes singular (goes to infinity) when  $1 - \sin x = 0$ , (i.e. when  $x = \frac{\pi}{2} + 2\pi n$  for integers  $n$ ). Therefore, our antiderivative function will be truly so on any interval which does not contain  $\pm\frac{\pi}{2} + 2\pi n$  for any integer  $n$ . One such domain would be the open interval,

$$(-\pi/2, \pi/2).$$