**2A-14*** Suppose that a piece of bubble gum has volume 4 cubic centimeters.

a) Use a linear approximation to calculate the thickness of a bubble of inner radius 10 centimeters.

(Start with the relation between the volume $V$ of a sphere and the radius $r$, and derive the approximate relation between $\Delta V$ and $\Delta r$.)

b) Find the exact answer.

c) To how many significant figures is the linear approximation accurate? In other words, find the order of magnitude of the difference between the approximation and the exact answer. (Be sure you use enough digits of $\pi$ to reflect correctly this accuracy!)

d) Use a quadratic approximation to the exact formula for the thickness that you found in part (b) to get an even more accurate estimate.

e) Why is the quadratic term comparable to the error in the accuracy of the linear approximation?

**2A-15** Find the linear and quadratic approximations to $\cos(3x)$ near $x = 0$, $\pi/6$, and $\pi/3$.

**2A-16** a) Use the law of cosines to find the formula $\sqrt{2 - \cos(2\pi/n)}$ for the length of the side of the equilateral $n$-gon inscribed in the unit circle.

b) Compute the perimeter and then compute the limit as $n$ tends to infinity using the quadratic approximation to $\cos\theta$ near $\theta = 0$. (Compare with 1J-4.)

## 2B. Curve Sketching

**2B-1** Sketch the graphs of the following. Find the intervals on which it is increasing and decreasing and decide how many solutions there are to $y = 0$. (Graphs need not reflect inflection points, which are discussed in 9-2).

a) $y = x^3 - 3x + 1$

b) $y = x^4 - 4x + 1$

c) $y'(x) = 1/(1 + x^2)$ and $y(0) = 0$.

d) $y = x^2/(x - 1)$

e) $y = x/(x + 4)$

f) $y = \sqrt{x+1}/(x - 3)$

g) $y = 3x^4 - 16x^3 + 18x^2 + 1$

h) $y = e^{-x^2}$

i) $y' = e^{-x^2}$ and $y(0) = 0$.

**2B-2** Find the inflection points of the graphs in problem 1.

**2B-3** Find the conditions on $a$, $b$ and $c$ for which the cubic

$$y = x^3 + ax^2 + bx + c$$

has a local maximum and a local minimum. Use the following two methods:

a) Find the condition under which $y'$ has two distinct real roots. Which of these roots is at the local maximum and which is at the local minimum? (Draw a picture.)

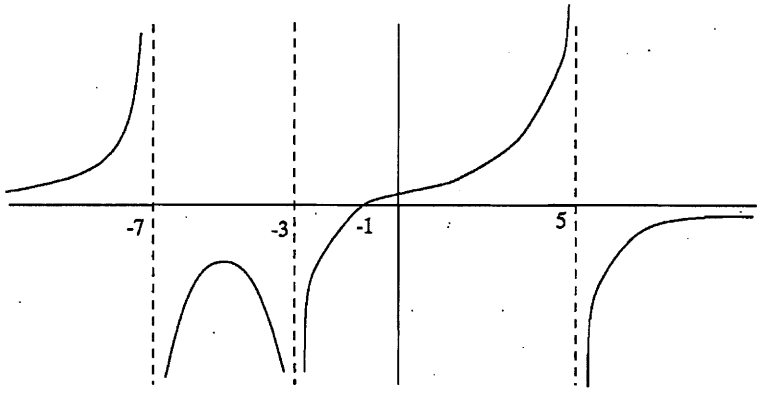
b) Find the condition under which $y' < 0$ at the inflection point. Why does this property imply that there is a local maximum and a local minimum?

**2B-4** Suppose that $f$ is a continuous function on $0 \le x \le 10$. Sketch the graph from the following description: $f$ is zero at 4, 7, and 9. $f'(x) > 0$ on $0 < x < 5$ and $8 < x < 10$

and $f'(x) < 0$ on $5 < x < 8$. With the given information, can you say anything for certain about the maximum value, the minimum value of $f$? Can you say anything about the place where the maximum is attained or the place where the minimum is attained?

**2B-5** a) Trace a copy of the graph of the function below and draw the graph of the derivative directly underneath. Connect the inflection point to the corresponding point on the graph of the derivative with a vertical dotted line.

b) Find a rational function with a graph resembling the one below.




**2B-6** a) Find a cubic polynomial with a local maximum at $x = -1$ and a local minimum at $x = 1$.

b) Draw the graph of the cubic on $-3 \leq x \leq 3$.

c) Draw a differentiable function on $-3 \leq x \leq 3$ that has an absolute maximum at $x = -1$ and an absolute minimum at $x = 1$.

**2B-7** a) Prove that if $f(x)$ is increasing and it has a derivative at a, then $f'(a) \geq 0$. (You may use the fact that a positive function has a limit $\geq 0$.)

b) If the conclusion of part (a) is changed to : $f'(a) > 0$, the statement becomes false. Indicate why the proof of part (a) fails to show that $f'(a) > 0$, and give a counterexample to the conclusion $f'(a) > 0$ (i.e., an example for which it is false).

c) Prove that if $f(x)$ has relative maximum at $a$ and it has a derivative at $a$, then $f'(a) = 0$. (Consider the right-and left-hand limits of $\Delta y/\Delta x$; apply the ideas of part (a).)

## ~~2C. Max-min problems~~

~~**2C-1** Cut four identical squares out of the corners of a 12 by 12 inch piece of cardboard and fold the sides so as to make a box without a top. Find the size of the corner square that maximizes the volume of the box.~~

~~**2C-2** You are asked to design a rectangular barnyard enclosing 20,000 square feet with fencing on three sides and the wall of a long barn on the fourth. Find the shortest length of fence needed.~~

~~**2C-3** What is the largest value of the product $xy$, if $x$ and $y$ are related by $x + 2y = a$, where $a$ is a fixed positive constant?~~

**2A-16** a) The law of cosines says that for a triangle with sides $a$, $b$, and $c$, with $\theta$ opposite the side of length $c$,

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Apply it to one of the $n$ triangles with vertex at the origin: $a = b = 1$ and $\theta = 2\pi/n$. So the formula is

$$c = \sqrt{2 - 2\cos(2\pi/n)}$$

b) The perimeter is $n\sqrt{2 - 2\cos(2\pi/n)}$. The quadratic approximation to $\cos\theta$ near 0 is

$$\cos\theta \approx 1 - \theta^2/2$$

Therefore, as $n \to \infty$ and $\theta = 2\pi/n \to 0$,

$$n\sqrt{2 - 2\cos(2\pi/n)} \approx n\sqrt{2 - 2(1 - (1/2)(2\pi/n)^2)} = n\sqrt{(2\pi/n)^2} = n(2\pi/n) = 2\pi$$

In other words,

$$\lim_{n\to\infty} n\sqrt{2 - 2\cos(2\pi/n)} = 2\pi,$$

the circumference of the circle of radius 1.

## 2B. Curve Sketching

**2B-1** a) $y = x^3 - 3x + 1$, $y' = 3x^2 - 3 = 3(x-1)(x+1)$. $y' = 0 \implies x = \pm 1$.

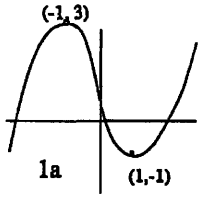
Endpoint values: $y \to -\infty$ as $x \to -\infty$, and $y \to \infty$ as $x \to \infty$.

Critical values: $y(-1) = 3$, $y(1) = -1$.

Increasing on: $-\infty < x < -1$, $1 < x < \infty$.

Decreasing on: $-1 < x < 1$.

Graph: $(-\infty, -\infty) \nearrow (-1, 3) \searrow (1, -1) \nearrow (\infty, \infty)$, crossing the $x$-axis three times.




b) $y = x^4 - 4x + 1$, $y' = x^3 - 4$. $y' = 0 \implies x = 4^{1/3}$.

Increasing on: $4^{1/3} < x < \infty$; decreasing on: $-\infty < x < 4^{1/3}$.

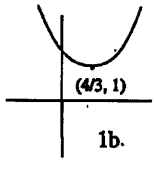
Endpoint values: $y \to \infty$ as $x \to \pm\infty$; critical value: $y(4^{1/3}) = 1$.

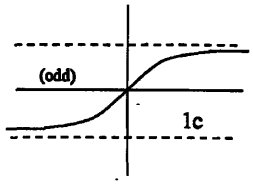
Graph: $(-\infty, \infty) \searrow (4^{1//3}, 1) \nearrow (\infty, \infty)$, never crossing the $x$-axis. (See below.)

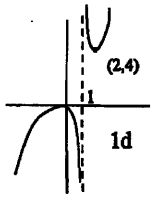
c) $y'(x) = 1/(1 + x^2)$ and $y(0) = 0$. By inspection, $y' > 0$ for all $x$, hence always increasing.

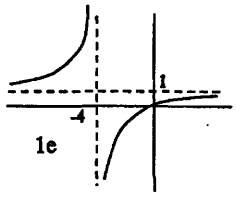
Endpoint values: $y \to c$ as $x \to \infty$ and by symmetry $y \to -c$ as $x \to -\infty$. (But it is not clear at this point in the course whether $c = \infty$ or some finite value. It turns out (in Lecture 26) that $y \to c = \pi/2$.

Graph: $(-\infty, -c) \nearrow (\infty, c)$, crossing the $x$-axis once (at $x = 0$). (See below.)



d) $y = x^2/(x-1)$, $y' = (2x(x-1)-x^2)/(x-1)^2 = (x^2-2x)/(x-1)^2 = (x-2)x/(x-1)^2$.

Endpoint values: $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$.

**Singular values:** $y(1^+) = +\infty$ and $y(1^-) = -\infty$.

Critical values: $y(0) = 0$ and $y(2) = 4$.

**New feature:** Pay attention to sign changes in the denominator of $y'$.

Increasing on: $-\infty < x < 0$ and $2 < x < \infty$

Decreasing on: $0 < x < 1$ and $1 < x < 2$

Graph: $(-\infty, -\infty) \nearrow (0,0) \searrow (1, -\infty) \uparrow (1, \infty) \searrow (2,4) \nearrow (\infty, \infty)$, crossing the $x$-axis once (at $x = 0$).

*Commentary on singularities: Look out for sign changes both where $y'$ is zero and also where $y'$ is undefined: $y' = 0$ indicates a possible sign change in the numerator and $y'$ undefined indicates a possible sign change in the denominator. In this case there was no sign change in $y'$ at $x = 1$, but there would have been a sign change, if there had been an odd power of $(x-1)$ in the denominator.*

e) $y = x/(x+4)$, $y' = ((x+4)-x)/(x+4)^2 = 4/(x+4)^2$. No critical points.

Endpoint values: $y \to 1$ as $x \to \pm\infty$.

Increasing on: $-4 < x < \infty$.

Decreasing on: $-\infty < x < -4$.

Singular values: $y(-4^+) = -\infty$, $y(-4^-) = +\infty$.

Graph: $(-\infty, 1) \nearrow (-4, \infty) \downarrow (-4, -\infty) \nearrow (\infty, 1)$, crossing the $x$-axis once (at $x = 0$).

f) $y = \sqrt{x+1}/(x-3)$, $y' = -(1/2)(x+5)(x+1)^{-1/2}(x-3)^{-2}$ No critical points because $x = -5$ is outside of the domain of definition, $x \geq -1$.

Endpoint values: $y(-1) = 0$, and as $x \to \infty$,

$$y = \frac{1 + \frac{1}{x}}{\sqrt{x} - \frac{3}{\sqrt{x}}} \to \frac{1}{\infty} = 0$$

Singular values: $y(3^+) = +\infty$, $y(3^-) = -\infty$.

Increasing on: nowhere

Decreasing on: $-1 < x < 3$ and $3 < x < \infty$.

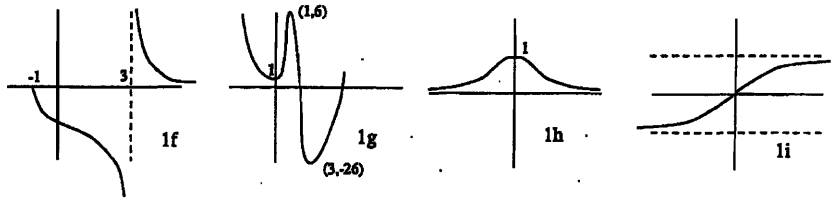
Graph: $(-1, 0) \searrow (3, -\infty) \uparrow (3, \infty) \searrow (\infty, 0)$, crossing the $x$-axis once (at $x = -1$).

g) $y = 3x^4 - 16x^3 + 18x^2 + 1$, $y' = 12x^3 - 48x^2 + 36x = 12x(x-1)(x-3)$. $y' = 0 \implies x = 0, 1, 3$.

Endpoint values: $y \to \infty$ as $x \to \pm\infty$.

Critical values: $y(0) = 1$, $y(1) = 6$, and $y(3) = -188$.

Increasing on: $0 < x < 1$ and $3 < x < \infty$.




Decreasing on: $-\infty < x < 0$ and $1 < x < 3$.

Graph: $(-\infty, \infty) \searrow (0,1) \nearrow (1,6) \searrow (3,-188) \nearrow (\infty, \infty)$, crossing the $x$-axis once.

h) $y = e^{-x^2}$, $y' = -2xe^{-x^2}$. $y' = 0 \implies x = 0$.

Endpoint values: $y \to 0$ as $x \to \pm\infty$.

Critical value: $y(0) = 1$.

Increasing on: $-\infty < x < 0$

Decreasing on: $0 < x < \infty$

Graph: $(-\infty, 0) \nearrow (0,1) \searrow (\infty, 0)$, never crossing the $x$-axis. (The function is even.)

i) $y' = e^{-x^2}$ and $y(0) = 0$. Because $y'$ is even and $y(0) = 0$, $y$ is odd. No critical points.

Endpoint values: $y \to c$ as $x \to \infty$ and by symmetry $y \to -c$ as $x \to -\infty$. It is not clear at this point in the course whether $c$ is finite or infinite. But we will be able to show that $c$ is finite when we discuss improper integrals in Unit 6. (Using a trick with iterated integrals, a subject in 18.02, one can show that $c = \sqrt{\pi}/2$.)

Graph: $(-\infty, -c) \nearrow (\infty, c)$, crossing the $x$-axis once (at $x = 0$).

**2B-2** a) One inflection point at $x = 0$. ($y'' = 6x$)

b) No inflection points. $y'' = 3x^2$, so the function is convex. $x = 0$ is not a point of inflection because $y'' > 0$ on both sides of $x = 0$.

c) Inflection point at $x = 0$. ($y'' = -2x/(1+x^2)^2$)

d) No inflection points. Reasoning: $y'' = 2/(x-1)^3$. Thus $y'' > 0$ and the function is concave up when $x > 1$, and $y'' < 0$ and the function is concave down when $x < 1$. But $x = 1$ is not called an inflection point because the function is not continuous there. In fact, $x = 1$ is a singular point.

e) No inflection points. $y'' = -8/(x+1)^3$. As in part (d) there is a sign change in $y''$, but at a singular point not an inflection point.

f) $y'' = -(1/2)[(x+1)(x-3) - (1/2)(x+5)(x-3) - 2(x+5)(x+1)](x+1)^{-3/2}(x-3)^3$

$= -(1/2)[-(3/2)x^2 - 15x - 11/2](x+1)^{-3/2}(x-3)^3$

Therefore there are two inflection points, $x = (-30 \pm \sqrt{768})/6$, $\approx 9.6, .38$.

g) $y'' = 12(3x^2 - 8x + 36)$. Therefore there are no inflection points. The quadratic equation has no real roots.

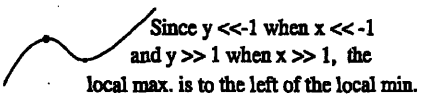
h) $y'' = (-2 + 4x^2)e^{-x^2}$. Therefore there are two inflection points at $x = \pm 1/\sqrt{2}$.

i) One inflection point at $x = 0$. ($y'' = -2xe^{-x^2}$)

**2B-3** a) $y' = 3x^2 + 2ax + b$. The roots of the quadratic polynomial are distinct real numbers if the discriminant is positive. (The *discriminant* is defined as the number under the square root in the quadratic formula.) Therefore there are distinct real roots if and only if

$$(2a)^2 - 4(3)b > 0, \qquad \text{or} \qquad a^2 - 3b > 0.$$

From the picture, since $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$, the larger root of $3x^2 + 2ax + b = 0$ (with the plus sign in the quadratic formula) must be the local min, and the smaller root must be the local max.




b) $y'' = 6x + 2a$, so the inflection point is at $-a/3$. Therefore the condition $y' < 0$ at the inflection point is

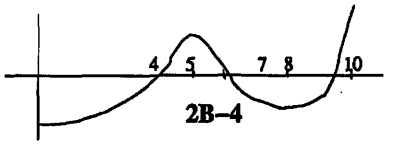
$$y'(-a/3) = 3(-a/3)^2 + 2a(-a/3) + b = -a^2/3 + b < 0,$$

which is the same as

$$a^2 - 3b > 0.$$

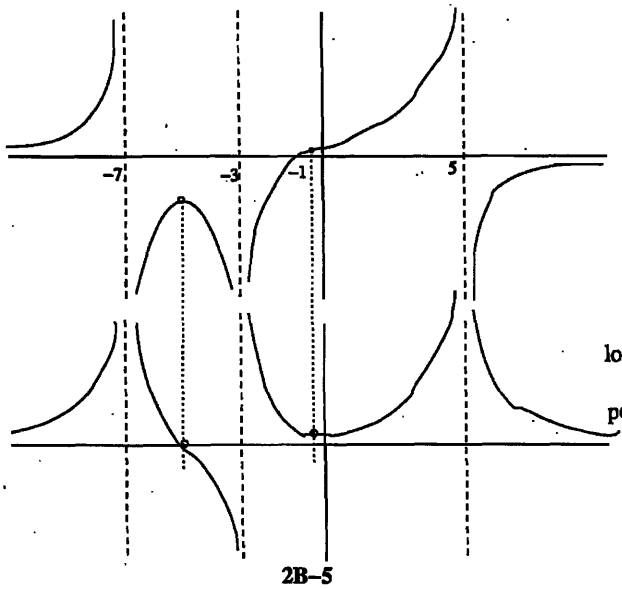
If $y' < 0$ at some point $x_0$, then the function is decreasing at that point. But $y \to \infty$ as $x \to \infty$, so there must be a local minimum at a point $x > x_0$. Similarly, since $y \to -\infty$ as $x \to -\infty$, there must be a local maximum at a point $x < x_0$.

**Comment:** We evaluate $y'$ at the inflection point of $y$ ($x = -a/3$) since we are trying to decide (cf. part (b)) whether $y'$ is ever negative. To do this, we find the minimum of $y'$ (which occurs where $y'' = 0$).




2B–4

Max is at x = 5 or x = 10;
Min is at x = 0 or x = 8.




2B–5

Graph of function

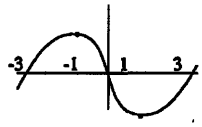
Graph of derivative; note that
local maximum point above corresponds
to zero below;
point of inflection above corresponds to
local minimum below.

**2B-6** a) Try $y' = (x+1)(x-1) = x^2 - 1$. Then $y = x^3/3 - x + c$. The constant $c$ won't matter so set $c = 0$. It's also more convenient to multiply by 3:

$$y = x^3 - 3x$$

b) This is an odd function with local min and max: $y(1) = -2$ and $y(-1) = 2$. The endpoints values are $y(3) = 18$ and $y(-3) = -18$. It is very steep: $y'(3) = 8$ .

c)




**2B-7** a) $f'(a) = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$

If y increasing then $\left.\begin{array}{l} \Delta y > 0 \Rightarrow \Delta x > 0 \\ \Delta y < 0 \Rightarrow \Delta x < 0 \end{array}\right\}$. So in both cases $\dfrac{\Delta y}{\Delta x} > 0$.

Therefore, $\lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} \geq 0$.

b) Proof breaks down at the last step. Namely, $\dfrac{\Delta y}{\Delta x} > 0$ doesn't imply $\lim_{\Delta x \to 0} \dfrac{\Delta x}{\Delta y} > 0$

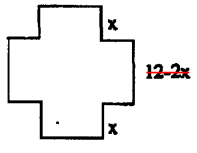
[Limits don't preserve strict inequalities, only weak ones. For example, $u^2 > 0$ for $u \neq 0$, but $\lim_{u \to 0} u^2 = 0 \geq 0$, not $> 0$.]

Counterexample: $f(x) = x^3$ is increasing for all x, but $f'(0) = 0$.

c) Use $f(a) \geq f(x)$ to show that $\lim_{\Delta x \to 0+} \Delta y/\Delta x \leq 0$ and $\lim_{\Delta x \to 0-} \Delta y/\Delta x \geq 0$. Since the left and right limits are equal, the derivative must be zero.

## ~~2C. Max-min problems~~

~~**2C-1** The base of the box has sidelength $12 - 2x$ and the height is $x$, so the volume is $V = x(12-2x)^2$.~~

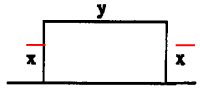



~~At the endpoints $x = 0$ and $x = 6$, the volume is 0, so the maximum must occur in between at a critical point.~~

$$V' = (12-2x)^2 + x(2)(12-2x)(-2) = (12-2x)(12-2x-4x) = (12-2x)(12-6x).$$

~~It follows that $V' = 0$ when $x = 6$ or $x = 2$. At the endpoints $x = 0$ and $x = 6$ the volume is 0, so the maximum occurs when $x = 2$.~~

~~**2C-2** We want to minimize the fence length $L = 2x + y$, where the variables $x$ and $y$ are related by $xy = A = 20,000$.~~




~~Choosing $x$ as the independent variable, we have $y = A/x$, so that $L = 2x + A/x$. At the endpoints $x = 0$ and $x = \infty$ (it's a long barn), we get $L = \infty$, so the minimumof $L$ must occur at a critical point.~~

$$L' = 2 - \frac{A}{x^2}; \quad L = 0 \Longrightarrow x^2 = \frac{A}{2} = 10,000 \Longrightarrow x = 100 \text{ feet}$$